Summary from previous lecture

• Laplace transform

\[ \mathcal{L} [f(t)] \equiv F(s) = \int_{0^{-}}^{\infty} f(t) e^{-st} \, dt. \]

\[ \mathcal{L} [u(t)] \equiv U(s) = \frac{1}{s}. \]

\[ \mathcal{L} [e^{-at}] = \frac{1}{s + a}. \]

\[ \mathcal{L} [\dot{f}(t)] = sF(s) - f(0^{-}). \]

\[ \mathcal{L} \left[ \int_{0^{-}}^{t} f(\xi) \, d\xi \right] = \frac{F(s)}{s}. \]

• Transfer functions and impedances

\[ T_F(s) = \frac{X(s)}{F(s)} \]

\[ Z(s) = \frac{F(s)}{X(s)} \]

\[ Z_J = Js; \quad Z_b = b; \quad \text{TF}(s) = \frac{1}{Z_J + Z_b} \]
Goals for today

- Dynamical variables in electrical systems:
  - charge,
  - current,
  - voltage.
- Electrical elements:
  - resistors,
  - capacitors,
  - inductors,
  - amplifiers.
- Transfer Functions of electrical systems (networks)
- **Next lecture (Friday):**
  - DC motor (electro-mechanical element) model
  - DC motor Transfer Function
Electrical dynamical variables: charge, current, voltage

- **Charge**: $q$
- **Charge flow** $\equiv$ **Current**: $i(t)$
- **Voltage** (aka potential): $v(t)$

**Units:**
- **Coulomb** [Cb]
- **Ampère** [A] = [Cb]/[sec]
- **Volt** [V]

Mathematically:

$$i(t) := \frac{dq(t)}{dt}$$
Electrical resistance

\[ v(t) = Ri(t) \Rightarrow V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R = Z_R \]

- Collisions between the mobile charges and the material fabric (ions, generally disordered) lead to energy dissipation (loss). As result, energy must be expended to generate current along the resistor; i.e., the current flow requires application of potential across the resistor.

- The quantity \( Z_R = R \) is called the resistance (unit: Ohms, or \( \Omega \))

- The quantity \( G_R = \frac{1}{R} \) is called the conductance (unit: Mhos or \( \Omega^{-1} \))
Capacitance

- Since similar charges repel, the potential \( v \) is necessary to prevent the charges from flowing away from the electrodes (discharge).
- Each change in potential \( v(t+\Delta t)=v(t)+\Delta v \) results in change of the energy stored in the capacitor, in the form of charges moving to/away from the electrodes (\( \leftrightarrow \) change in electric field).
Capacitance

\[ q(t) = Cv(t) \Rightarrow \frac{dq(t)}{dt} \equiv i(t) = C\frac{dv(t)}{dt} \]

- Capacitance \( C \):

- in Laplace domain: \( I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} \equiv Z_C(s) = \frac{1}{Cs} \)
Inductance

- Current flow \( i \) around a loop results in magnetic field \( B \) pointing normal to the loop plane. The magnetic field counteracts changes in current; therefore, to effect a change in current \( i(t + \Delta t) = i(t) + \Delta i \) a potential \( v \) must be applied (i.e., energy expended).

- Inductance \( L \):
  \[
  v(t) = L \frac{di(t)}{dt}
  \]

- In Laplace domain:
  \[
  V(s) = LsI(s) \Rightarrow \frac{V(s)}{I(s)} = Z_L(s) = Ls
  \]
Summary: passive electrical elements; Sources

Table removed due to copyright restrictions.

Combining electrical elements: networks

Network analysis relies on two physical principles

- **Kirchhoff Current Law (KCL)**
  - charge conservation

\[
\sum i_k(t) = 0
\]

- **Kirchhoff Voltage Law (KVL)**
  - energy conservation

\[
\sum V_k(s) = 0
\]
Impedances in series and in parallel

**Impedances in series**

KCL: $I_1 = I_2 \equiv I$.

KVL: $V = V_1 + V_2$.

From definition of impedances:

$$Z_1 = \frac{V_1}{I_1}; \quad Z_2 = \frac{V_2}{I_2}.$$  

Therefore, equivalent circuit has

$$Z = Z_1 + Z_2 \left( \Leftrightarrow \frac{1}{G} = \frac{1}{G_1} + \frac{1}{G_2}. \right)$$

**Impedances in parallel**

KCL: $I = I_1 + I_2$.

KVL: $V = V_1 + V_2 \equiv V$.

From definition of impedances:

$$Z_1 = \frac{V_1}{I_1}; \quad Z_2 = \frac{V_2}{I_2}.$$  

Therefore, equivalent circuit has

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \left( \Leftrightarrow G = G_1 + G_2. \right)$$
The voltage divider

Since the two impedances are in series, they combine to an equivalent impedance

\[ Z = Z_1 + Z_2. \]

The current flowing through the combined impedance is

\[ I = \frac{V}{Z}. \]

Therefore, the voltage drop across \( Z_2 \) is

\[ V_2 = Z_2I = Z_2 \frac{V}{Z} \Rightarrow \frac{V_2}{V_i} = \frac{Z_2}{Z_1 + Z_2}. \]
Example: the **RC circuit**

\[ Z_1 = R \]

\[ Z_2 = \frac{1}{Cs} \]

\[ V_{i} \]

\[ V_{C} \]

We recognize the voltage divider configuration, with the voltage across the capacitor as output. The transfer function is obtained as

\[ \text{TF}(s) = \frac{V_{C}(s)}{V_{i}(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + RCs} = \frac{1}{1 + \tau s}, \]

where \( \tau = RC \). Further, we note the similarity to the transfer function of the rotational mechanical system consisting of a motor, inertia \( J \) and viscous friction coefficient \( b \) that we saw in Lecture 3. [The transfer function was \( 1/b(1 + \tau s) \), *i.e.* identical within a multiplicative constant, and the time constant \( \tau \) was defined as \( J/b \).] We can use the analogy to establish properties of the RC system without re-deriving them: *e.g.*, the response to a step input \( V_{i} = V_{0}u(t) \) (step response) is

\[ V_{C}(t) = V_{0} \left(1 - e^{-t/\tau}\right) u(t), \quad \text{where now } \tau = RC. \]
Interpretation of the **RC** step response

\[ Z_1 = R \]
\[ Z_2 = \frac{1}{C_s} \]

\[ V_C(t) = V_0 \left(1 - e^{-t/\tau}\right) u(t), \quad \tau = RC. \]

\[ V_0 = 1 \text{ Volt} \quad R = 2k\Omega \quad C = 1\mu F \]

Charging of a capacitor: becomes progressively more difficult as charges accumulate. Capacity (steady-state) is reached asymptotically \((V_C \rightarrow V_0 \text{ as } t \rightarrow \infty)\).
Example: RLC circuit with voltage source

\[ V(s) = \frac{1}{LC} \left( s^2 + \frac{R}{L}s + \frac{1}{LC} \right) \]

\[ V_C(s) \]

Figure by MIT OpenCourseWare.
Example: two-loop network

Images removed due to copyright restrictions.

Please see: Fig. 2.6 and 2.7 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.
The operational amplifier (op-amp)

(a) Generally, \( v_o = A (v_2 - v_1) \), where \( A \) is the amplifier gain.

(b) When \( v_2 \) is grounded, as is often the case in practice, then \( v_o = -Av_1 \). (Inverting amplifier.)

(c) Often, \( A \) is large enough that we can approximate \( A \rightarrow \infty \).

Rather than connecting the input directly, the op-amp should then instead be used in the feedback configuration of Fig. (c).

We have:

\[
V_1 = 0; \quad I_a = 0
\]

(because \( V_o \) must remain finite) therefore

\[
I_1 + I_2 = 0;
\]

\[
V_i - V_1 = V_i = I_1 Z_1;
\]

\[
V_o - V_1 = V_o = I_2 Z_2.
\]

Combining, we obtain

\[
\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}.
\]