Last week: analysis of pinion-rack with velocity feedback

Calculation of the steady-state error

Transfer function:
\[
\frac{V(s)}{V_{ref}(s)} = \frac{0.3162K}{s + 2 + 0.3162K}
\]

Step input: \( V_{ref}(s) = \frac{1}{s} \)

Output: \( V(s) = \frac{1}{s} \times \frac{0.3162K}{s + 2 + 0.3162K} \)

Steady-state: \( v(\infty) = \lim_{s \to 0} sV(s) = \frac{0.3162K}{2 + 0.3162K} \)

Steady-state error: \( e(\infty) = 1 - v(\infty) = \frac{2}{2 + 0.3162K} \)

\[ e(\infty) \quad [\text{for } K = 100] \]
Today

• Analysis of steady-state errors
  – Definition for step, ramp, parabola inputs
  – Steady-state error in unity feedback systems
  – System type and static error constants
  – The role of integrators
  – Steady-state error in the presence of disturbances
  – Controller gain and step disturbance cancellation

• Wednesday & Friday: Root locus
Inserting an integrator

We’ve changed our mind!
Now the output is the rack position $x(t)$.

The feedback to the diff-amp must also change to measure position

Since $v(t) = \frac{dx(t)}{dt} \Leftrightarrow x(t) = \int_0^t v(t')dt'$

we don’t need to re-compute the plant TF.

$$X(s) = \frac{1}{s}V(s),$$

where $V(s)$ is the output velocity of the previous system that we have already analyzed.
Inserting an integrator

Calculation of the closed-loop TF

Plant TF: \( \frac{X(s)}{V_s(s)} = \frac{0.3162}{s(s+2)} \)

Closed loop TF: \( \frac{X(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2 + 2s + 0.3162K} \)

Calculation of the steady-state error

Step input: \( V_{ref}(s) = \frac{1}{s} \)

\[ x(\infty) = \lim_{s \to 0} s \times \frac{1}{s} \times \frac{0.3162K}{s^2 + 2s + 0.3162K} = 1. \]

\[ e(\infty) = 1 - x(\infty) = 0. \]
Generalizing: different system inputs

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Please see: Table 7.1 and Fig. 7.1 in Nise, Norman S. Control Systems Engineering, 4th ed. Hoboken, NJ: John Wiley, 2004.
Generalizing: steady-state error for arbitrary input

• Unit step input:
  Steady-state error = unit step – output as $t \to \infty$

• Ramp input:
  Steady-state error = ramp – output as $t \to \infty$

Generally, the steady-state error is defined as

$$e(\infty) = \lim_{t \to \infty} \left[ r(t) - c(t) \right] = \lim_{s \to 0} s \left[ R(s) - C(s) \right],$$

where the last equality follows from the final value theorem.
Generalizing: steady-state error for arbitrary system, unity feedback

From the definition of the steady-state error,

\[ e(\infty) = \lim_{s \to 0} s \left[ R(s) - C(s) \right] = \lim_{s \to 0} sE(s). \]

From the block diagram we can also see that

\[ \frac{C(s)}{E(s)} = G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}. \]
Generalizing: steady-state error for arbitrary system, unity feedback

Recall the closed-loop TF of the unity feedback system

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \Rightarrow C(s) = \frac{R(s)G(s)}{1 + G(s)}.
\]

Substituting into the two formulae from the previous page,

\[
E(s) = \frac{R(s)}{1 + G(s)} \Rightarrow e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}.
\]
Steady-state error and static error constants

\[ e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}. \]

\[ e(\infty) = \lim_{s \to 0} \frac{1}{s} \times \frac{s}{1 + G(s)} = \lim_{s \to 0} \frac{1}{1 + \lim_{s \to 0} G(s)} = \frac{1}{1 + K_p} \]

where \( K_p = \lim_{s \to 0} G(s) \).

\[ e(\infty) = \lim_{s \to 0} \frac{1}{s} \times \frac{s}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s + s G(s)} = \lim_{s \to 0} \frac{1}{s G(s)} = \frac{1}{1 + K_v} \]

where \( K_v = \lim_{s \to 0} s G(s) \).

\[ e(\infty) = \lim_{s \to 0} \frac{1}{s} \times \frac{s}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} = \lim_{s \to 0} \frac{1}{s^2 G(s)} = \frac{1}{1 + K_a} \]

where \( K_a = \lim_{s \to 0} s^2 G(s) \).

Note: the system must be \textbf{stable} (i.e., all poles on left-hand side or at the origin) for these calculations to apply.
System types and steady-state errors

\[ \mathcal{L}[u(t)] = \frac{1}{s} \]

\[ K_p = \lim_{s \to 0} G(s) \]

\[ \mathcal{L}[tu(t)] = \frac{1}{s^2} \]

\[ K_v = \lim_{s \to 0} sG(s) \]

\[ \mathcal{L}[\frac{1}{2}t^2u(t)] = \frac{1}{s^3} \]

\[ K_a = \lim_{s \to 0} s^2G(s) \]

Table removed due to copyright restrictions.
Disturbances

From the I–O relationship of the plant,

\[ C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s). \]

From the summation element,

\[ E(s) = R(s) - C(s). \]

Substituting \( C(s) \) and solving for \( E(s) \),

\[ E(s) = R(s) \frac{1}{1 + G_1(s)G_2(s)} - D(s) \frac{G_2(s)}{1 + G_1(s)G_2(s)} \]

Equivalent block diagram with \( D(s) \) as input and \( -E(s) \) as output.
Disturbances

\[ e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \left[ R(s) \frac{1}{1 + G_1(s)G_2(s)} - D(s) \frac{G_2(s)}{1 + G_1(s)G_2(s)} \right] \equiv e_R(\infty) + e_D(\infty), \]

where

\[ e_R(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G_1(s)G_2(s)} \]

\[ e_D(\infty) = -\lim_{s \to 0} \frac{sG_2(s)D(s)}{1 + G_1(s)G_2(s)}. \]
Unit step disturbance

Special case: unit step disturbance \( D(s) = \frac{1}{s} \).

\[ e_D(\infty) = -\lim_{s \to 0} \frac{sG_2(s) \times (1/s)}{1 + G_1(s)G_2(s)} \]
\[ = -\lim_{s \to 0} \frac{G_2(s)}{1 + \lim_{s \to 0} G_1(s)G_2(s)} \]
\[ = -\frac{1}{\lim_{s \to 0} G_2(s) + \lim_{s \to 0} G_1(s)} \].

If \( K_1, K_2 \) are the gains of the controller and plant, respectively, then

\[ e(\infty) \downarrow \text{ if } K_1 \uparrow \text{ or } K_2 \downarrow \].
Unit step disturbance: example

DC motor with rack–pinion load, position feedback,

subject to unit step disturbance \( D(s) = \frac{1}{s} \).

\[
e_D(\infty) = -\frac{1}{\lim_{s \to 0} G_2(s) + \lim_{s \to 0} G_1(s)}
= -\frac{1}{\lim_{s \to 0} \frac{s(s+2)}{0.3162} + \lim_{s \to 0} K}
= -\frac{1}{K}.
\]