**Problem Set 4**

**Assigned:** Oct. 2, 2008  
**Due:** Oct. 9, 2008

**Problem 1:** An AM (amplitude-modulated) radio signal \( f_{AM}(t) \) is described by

\[
f_{AM}(t) = (1 + a f_{audio}(t)) \sin(\Omega_c t)
\]

where \( f_{audio}(t) \) is the audio signal, \( \sin(\Omega_c t) \) is known as radio-frequency carrier signal (\( f_c = 500 - 1600 \) kHz - the AM band), and \( a \) is a positive constant that determines the modulation depth. (Note that we require \( |af_{audio}(t)| < 1 \) otherwise we have over-modulation.) The following figure shows an AM signal with an “audio” waveform that is a simple low frequency sinusoid. You can see how the audio signal “modulates” the amplitude of the rf signal.

(a) Sketch the magnitude of the Fourier transform of \( f_{AM}(t) \) when \( f_{audio}(t) = 0 \).

(b) Let \( a = 0.5 \), and sketch the magnitude of the Fourier transform of \( f_{AM}(t) \) when

\[
f_{audio}(t) = 0.5 \cos(2\pi \cdot 1000t) + 0.25 \cos(2\pi \cdot 2000t)
\]

(Hint: There is no need to actually compute the FT. Consider expanding \( f_{AM}(t) \), or simply use properties of the FT.)

(c) Use your result from (b) to generalize, and sketch the magnitude spectrum of \( f_{AM}(t) \) when \( f_{audio}(t) \) has a spectrum (again let \( a = 0.5 \):
(d) If $f_{\text{audio}}$ has a bandwidth $B = \Omega_u - \Omega_l$, what is the bandwidth of a band-pass filter that would be necessary to select the signal $f_{\text{AM}}(t)$ out of all the other AM radio stations?

**Problem 2:** We generally ignore in the phase response in filter design. Although you might wish for a “zero-phase” filter, you can see from the class handout on causality that a filter with a purely real frequency response is acausal, and as such cannot be implemented in a physical system. The following are a pair of tricks that may be used to do “off-line” zero-phase filtering of recorded data. (Note: these methods are used frequently in digital signal processing - it is difficult to do this in continuous time.)

Assume that you have a filter $H(j\Omega)$ with arbitrary phase response $\angle H(j\Omega)$, and that your input signal is $f(t)$ is recorded on a tape-recorder that can be played forwards or backwards.

**Method (1)**

1. Pass $f(t)$ through the filter and record the output $g(t)$ on another tape recorder.
2. Play $g(t)$ backwards through the filter (that is the filter input is $g(-t)$) and record the output $x(t)$.
3. The filtered output is found by playing the $x(t)$ backwards, that is $y(t) = x(-t)$.

**Method (2)**

1. Pass $f(t)$ through the filter and record the output $g(t)$.
2. Reverse $f(t)$ so as to pass $f(-t)$ through the filter and generate $x(t)$.
3. Reverse $x(t)$ and sum with $g(t)$ to form the output $y(t) = g(t) + x(-t)$.

Show that both methods generate an overall filter that has no phase shift, and find the overall magnitude response $|H_{\text{eq}}(j\Omega)|$ in each case. Hint: $\mathcal{F}\{f(-t)\} = \tilde{F}(j\Omega)$.

**Problem 3:** Problem 5 in Problem Set 2 examined an all-pass filter with a transfer function

$$H(s) = \frac{s - a}{s + a} \quad a > 0$$

and you showed that this filter had a frequency response in which $|H(j\Omega)| = 1$ at all frequencies.

Design an op-amp based first-order all-pass filter of this form that will have a phase shift of $-90^\circ$ at a frequency of 50 Hz. (Consider a modified form of the 3 op-amp circuit described in the handout - noting that you only need a first-order system.) Find “appropriate” values
for all resistors and capacitors.

**Problem 4:** Consider the second-order bandpass filter with transfer function

\[ H_{bp}(s) = \frac{a_1 s}{s^2 + a_1 s + a_0}. \]

Many books on signal processing express this transfer function in terms of two parameters \( \Omega_p \) and \( Q \),

\[ H_{bp}(s) = \frac{\Omega_p s}{s^2 + \Omega_p s + \Omega_p^2} \]

where \( \Omega_p \) is the (approximate) peak frequency (center of the passband), and \( Q \) is known as the “quality” factor.

**Aside:** If you compare this to the classic second-order system description used in system dynamics and control, that is

\[ H_{bp}(s) = \frac{2 \zeta \Omega_n s}{s^2 + 2 \zeta \Omega_n s + \Omega_n^2} \]

where \( \Omega_n \) is the undamped natural frequency, and \( \zeta \) is the damping ratio, you can see that

\[ Q = \frac{1}{2 \zeta}. \]

In this problem we examine the relationship between \( Q \) and the -3dB bandwidth of the second-order filter. Consider the magnitude plot below:

Let \( \Omega_u \) and \( \Omega_l \) be the upper and lower -3db (0.707) response frequencies as shown, and let \( \Delta = \Omega_u - \Omega_l \) be the -3dB bandwidth.
(a) Show

\[ \Omega_a = \Omega_p \sqrt{1 + \frac{1}{2Q^2}} + \frac{1}{Q} \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \]

\[ \Omega_l = \Omega_p \sqrt{1 + \frac{1}{2Q^2} - \frac{1}{Q} \sqrt{1 + \left(\frac{1}{2Q}\right)^2}} \]

(b) Use these results to show that the -3dB bandwidth of the second-order filter is

\[ \Delta = \frac{\Omega_p}{Q} \]

**Hint:** Write \( \Delta = \sqrt{a + b} - \sqrt{a - b} \).

(c) Determine the transfer function of a second-order bandpass filter with a center frequency of 100 Hz., and a -3dB bandwidth of 10 Hz.

**Problem 5:** A sampling system takes samples at regular intervals \( \Delta T \). Assume we have a sinusoid

\[ y(t) = \sin \Omega t, \]

so that the samples are \( y(n\Delta T) = \sin n\Omega \Delta T \). We know that if the frequency \( \Omega \) is greater than the Nyquist frequency \( \Omega_N = \pi/\Delta T \), the sample set is aliased.

Now consider two sinusoids, one \( y(t) = \sin n\Omega_0 \Delta T \) with a frequency \( \Omega_0 \) that is below the Nyquist frequency, and another with frequency \( \Omega_1 \) above the Nyquist frequency.

(a) Assume \( y_1(t) = \sin \Omega_1 t \) where

\[ \Omega_1 = 2k\Omega_N - \Omega_0, \quad k = 1 \ldots \infty \]

is any positive integer.

Show that the sample sets are related by \( y_1(n\Delta T) = -y(n\Delta T) = -\sin(n\Omega_0 \Delta T) \),

(b) Repeat part (a) with

\[ \Omega_1 = 2k\Omega_N + \Omega_0, \quad k = 1 \ldots \infty \]

is any positive integer.

and show that in this case the sample sets are identical, that is \( y_2(n\Delta T) = y(n\Delta T) = \sin(n\Omega_0 \Delta T) \).

(c) Use the results of (a) and (b) to graphically demonstrate the concept of “frequency folding” of aliased sinusoids.

(d) A periodic waveform is written as a Fourier Series

\[ y(t) = 5 \sin(2\pi(25)t) + 2\sin(2\pi(75)t) + 3\sin(2\pi(125)t). \]

If the waveform is sampled at 100 samples/sec, determine the frequencies and amplitudes of the spectral components in the sampled waveform. (Hint: The results of parts (a) and (b) should help.)