Reading:
- Class Handout: *Introduction to the Operational Amplifier*
- Class Handout: *Op-amp Implementation of Analog Filters*

1 Operational-Amplifier Based State-Variable Filters

We saw in Lecture 8 that second-order filters may be implemented using the block diagram structure

and that a high-order filter may be implemented by cascading second-order blocks, and possibly a first-order block (if the filter order is odd).

We now look into a method for implementing this filter structure using operational amplifiers.

1.1 The Operational Amplifier

What is an operational amplifier? It is simply a very high gain electronic amplifier, with a pair of differential inputs. Its functionality comes about through the use of feedback around the amplifier, as we show below.
The op-amp has the following characteristics:

- It is basically a “three terminal” amplifier, with two inputs and an output. It is a differential amplifier, that is the output is proportional to the difference in the voltages applied to the two inputs, with very high gain $A$,

$$v_{out} = A(v_+ - v_-)$$

where $A$ is typically $10^4 - 10^5$, and the two inputs are known as the non-inverting ($v_+$) and inverting ($v_-$) inputs respectively. In the ideal op-amp we assume that the gain $A$ is infinite.

- In an ideal op-amp no current flows into either input, that is they are voltage-controlled and have infinite input resistance. In a practical op-amp the input current is in the order of pico-amps ($10^{-12}$) amp, or less.

- The output acts as a voltage source, that is it can be modeled as a Thevenin source with a very low source resistance.

The following are some common op-amp circuit configurations that are applicable to the active filter design method described here. (See the class handout for other common configurations).

**The Inverting Amplifier:**

In the configuration shown above we note

- Because the gain $A$ is very large, the voltage at the node designated *summing junction* is very small, and we approximate it as $v_- = 0$ — the so-called virtual ground assumption.

- We assume that the current $i_-$ into the inverting input is zero.

Applying Kirchoff’s Current law at the summing junction we have

$$i_1 + i_f = \frac{v_{in}}{R_1} + \frac{v_o}{R_f} = 0$$

from which

$$v_{out} = -\frac{R_f}{R_{in}} v_{in}$$
The voltage gain is therefore defined by the ratio of the two resistors. The term *inverting* amplifier comes about because of the sign change.

**The Inverting Summer:** The inverting amplifier may be extended to include multiple inputs:

As before we assume that the inverting input is at a virtual ground \( v_\approx 0 \) and apply Kirchhoff’s current law at the summing junction

\[ i_1 + i_2 + i_f = \frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{V_{out}}{R_f} = 0 \]

or

\[ v_{out} = -\left( \frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2 \right) \]

which is a weighted sum of the inputs.

The summer may be extended to include several inputs by simply adding additional input resistors \( R_i \), in which case

\[ v_{out} = -\sum_{i=1}^{n} \frac{R_f}{R_i} v_i \]

**The Integrator:** If the feedback resistor in the inverting amplifier is replaced by a capacitor \( C \) the amplifier becomes an integrator.

At the summing junction we apply Kirchhoff’s current law as before but the feedback current is now defined by the elemental relationship for the capacitor:

\[ i_{in} + i_f = \frac{v_{in}}{R_{in}} + C \frac{dv_{out}}{dt} = 0 \]
Then
\[ \frac{dv_{out}}{dt} = -\frac{1}{R_{in}C}v_{in} \]
or
\[ v_{out} = -\frac{1}{R_{in}C} \int_0^t v_{in} \, dt + v_{out}(0) \]

As above, the integrator may be extended to a summing configuration by simply adding extra input resistors:

\[ v_{out} = -\frac{1}{C} \int_0^t \left( \sum_{i=1}^n \frac{v_i}{R_i} \right) \, dt + v_{out}(0) \]
and if all input resistors have the same value \( R \)

\[ v_{out} = -\frac{1}{RC} \int_0^t \left( \sum_{i=1}^n v_i \right) \, dt + v_{out}(0) \]

1.2 A Three Op-Amp Second-Order State Variable Filter

A configuration using three op-amps to implement low-pass, high-pass, and bandpass filters directly is shown below:

Amplifiers \( A_1 \) and \( A_2 \) are integrators with transfer functions

\[ H_1(s) = -\left( \frac{1}{R_1C_1} \right) \frac{1}{s} \quad \text{and} \quad H_2(s) = -\left( \frac{1}{R_2C_2} \right) \frac{1}{s}. \]

Let \( \tau_1 = R_1C_1 \) and \( \tau_2 = R_2C_2 \). Because of the gain factors in the integrators and the sign inversions we have

\[ v_1(t) = -\tau_2 \frac{dv_2}{dt} \quad \text{and} \quad v_3(t) = \tau_1 \tau_2 \frac{d^2v_2}{dt^2}. \]

Amplifier \( A_3 \) is the summer. However, because of the sign inversions in the op-amp circuits we cannot use the elementary summer configuration described above. Applying Kirchoff’s Current Law at the non-inverting and inverting inputs of \( A_3 \) gives

\[ \frac{V_{in} - v_+}{R_5} + \frac{v_1 - v_+}{R_6} = 0 \quad \text{and} \quad \frac{v_3 - v_-}{R_4} + \frac{v_2 - v_-}{R_1} = 0. \]
Using the infinite gain approximation for the op-amp, we set \( v_- = v_+ \) and

\[
\frac{R_3}{R_3 + R_4} v_3 - \frac{R_5}{R_5 + R_6} v_1 + \frac{R_4}{R_3 + R_4} v_2 = \frac{R_6}{R_5 + R_6} V_{in},
\]

and substituting for \( v_1 \) and \( v_3 \) we generate a differential equation in \( v_2 \)

\[
\frac{d^2 v_2}{dt^2} + \left( \frac{1 + R_4/R_3}{\tau_1(1 + R_6/R_5)} \right) \frac{dv_2}{dt} + \left( \frac{R_4}{R_3 \tau_1 \tau_2} \right) v_2 = \left( \frac{1 + R_4/R_3}{\tau_1 \tau_2 (1 + R_5/R_6)} \right) V_{in}
\]

which corresponds to a low-pass transfer function

\[
H(s) = \frac{K_{lp} a_0}{s^2 + a_1 s + a_0}
\]

where

\[
a_0 = \left( \frac{R_4}{R_3} \right) \frac{1}{\tau_1 \tau_2}, \quad a_1 = \left( \frac{1 + R_4/R_3}{1 + R_6/R_5} \right) \frac{1}{\tau_1}, \quad K_{lp} = \frac{1 + R_3/R_4}{1 + R_5/R_6}
\]

**A Band-Pass Filter:** Selection of the output as the output of integrator \( A_1 \) generates the transfer function

\[
H_{bp}(s) = -\tau_1 s H_{lp}(s) = \frac{-K_{bp} a_1 s}{s^2 + a_1 s + a_0}
\]

where

\[
K_{bp} = \frac{R_6}{R_5}
\]

**A High-Pass Filter:** Selection of the output as the output of the summer \( A_3 \) generates the transfer function

\[
H_{hp}(s) = \tau_1 \tau_2 s^2 H_{lp}(s) = \frac{K_{hp} s^2}{s^2 + a_1 s + a_0}
\]

where

\[
K_{hp} = \frac{1 + R_4/R_3}{1 + R_5/R_6}
\]

**A Band-Stop Filter:** The band-stop configuration may be implemented with an additional summer to add the outputs of amplifiers \( A_2 \) and \( A \) (with appropriate weights).

### 1.3 A Simplified Two Op-amp Based State-variable Filter:

If the required filter does not require a high-pass action (that is, access to the output of the summer \( A_1 \) above) the summing operation may be included at the input of the first integrator, leading to a simplified circuit using only two op-amps shown below:
Consider the input stage:

With the infinite gain assumption for the op-amps, that is \( V_- = V_+ \), and with the assumption that no current flows in either input, we can apply Kirchoff’s Current Law (KCL) at the node designated (a) above:

\[
i_1 + i_f - i_3 = (V_{in} - v_a) \frac{1}{R_1} + sC_1(v_1 - v_a) - v_a \frac{1}{R_3} = 0
\]

Assuming \( v_a = V_{out} \), and realizing that the second stage is a classical op-amp integrator with transfer function

\[
\frac{V_{out}(s)}{v_1(s)} = -\frac{1}{R_3C_2s}
\]

\[
(V_{in} - V_{out}) \frac{1}{R_1} + sC_1(-R_2C_2sV_{out} - V_{out}) - V_{out} \frac{1}{R_3} = 0
\]

which may be rearranged to give the second-order transfer function

\[
\frac{V_{out}(s)}{V_{in}(s)} = \frac{1/\tau_1\tau_2}{s^2 + (1/\tau_2)s + (1 + R_1/R_3)/\tau_1\tau_2}
\]

which is of the form

\[
H_{ip}(s) = \frac{K_{ip}a_0}{s^2 + a_1s + a_0}
\]

where

\[
a_0 = (1 + R_1/R_3) \frac{1}{\tau_1\tau_2}
\]

\[
a_1 = \frac{1}{\tau_2}
\]

\[
K_{ip} = \frac{1}{1 + R_1/R_3}
\]

9–6
1.4 First-Order Filter Sections:

Single pole low-pass filter sections with a transfer function of the form

\[ H(s) = \frac{K\Omega_0}{s + \Omega_0} \]

may be implemented in either an inverting or non-inverting configuration as shown in Fig. 11.

The inverting configuration (a) has transfer function

\[ \frac{V_{out}(s)}{V_{in}(s)} = - \frac{Z_f}{Z_{in}} = - \left( \frac{R_1}{R_2} \right) \frac{1/R_1C}{s + 1/R_1C} \]

where \( \Omega_0 = 1/R_1C \) and \( K = -R_1/R_2 \).

The non-inverting configuration (b) is a first-order R-C lag circuit buffered by a non-inverting (high input impedance) amplifier (see the class handout) with a gain \( K = 1 + R_3/R_2 \). Its transfer function is

\[ \frac{V_{out}(s)}{V_{in}(s)} = \left( 1 + \frac{R_3}{R_2} \right) \frac{1/R_1C}{s + 1/R_1C} \]
Classroom Demonstration

Example 2 in the class handout “Op-Amp Implementation of Analog Filters” describes a state-variable design for a 5th-order Chebyshev Type I low-pass filter with \( \Omega_c = 1000 \text{ rad/s} \) and 1dB ripple in the passband. The transfer function is

\[
H(s) = \frac{122828246505000}{(s^2 + 468.4s + 429300)(s^2 + 178.9s + 988300)(s + 289.5)}
\]

\[
= \frac{429300}{s^2 + 468.4s + 429300} \times \frac{988300}{s^2 + 178.9s + 988300} \times \frac{289.5}{s + 289.5}
\]

which is implemented in the handout as a pair of second-order two-op-amp sections followed by a first-order block:

This filter was constructed on a bread-board using 741 op-amps, and was demonstrated to the class, driven by a sinusoidal function generator and with an oscilloscope to display the output. The demonstration included showing (1) the approximately 10% ripple in the passband, and (2) the rapid attenuation of inputs with frequency above 157 Hz (1000 rad/s).
2 Introduction to Discrete-Time Signal Processing

Consider a continuous function \( f(t) \) that is limited in extent, \( T_1 \leq t < T_2 \). In order to process this function in the computer it must be sampled and represented by a finite set of numbers. The most common sampling scheme is to use a fixed sampling interval \( \Delta T \) and to form a sequence of length \( N \): \( \{ f_n \} \) \( (n = 0 \ldots N - 1) \), where

\[
  f_n = f(T_1 + n\Delta T).
\]

In subsequent processing the function \( f(t) \) is represented by the finite sequence \( \{ f_n \} \) and the sampling interval \( \Delta T \).

In practice, sampling occurs in the time domain by the use of an analog-digital (A/D) converter.

(i) The sampler (A/D converter) records the signal value at discrete times \( n\Delta T \) to produce a sequence of samples \( \{ f_n \} \) where \( f_n = f(n\Delta T) \) (\( \Delta T \) is the sampling interval).

(ii) At each interval, the output sample \( y_n \) is computed, based on the history of the input and output, for example

\[
  y_n = \frac{1}{3} (f_n + f_{n-1} + f_{n-2})
\]

3-point moving average filter, and

\[
  y_n = 0.8y_{n-1} + 0.2f_n
\]

is a simple recursive first-order low-pass digital filter. Notice that they are algorithms.

(iii) The reconstructor takes each output sample and creates a continuous waveform.

In real-time signal processing the system operates in an infinite loop:
2.1 Sampling

The mathematical operation of sampling (not to be confused with the operation of an analog-digital converter) is most commonly described as a multiplicative operation, in which \( f(t) \) is multiplied by a Dirac comb sampling function \( s(t; \Delta T) \), consisting of a set of delayed Dirac delta functions:

\[
    s(t; \Delta T) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T).
\]

We denote the sampled waveform \( f^*(t) \) as

\[
    f^*(t) = s(t; \Delta T) f(t) = \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T)
\]
Note that \( f^*(t) \) is a set of delayed and weighted delta functions, and that the waveform must be interpreted in the *distribution* sense by the strength (or area) of each component impulse. The implied process to produce the discrete sample sequence \( \{f_n\} \) is by integration across each impulse, that is

\[
f_n = \int_{n\Delta t-}^{n\Delta t+} f^*(t) dt = \int_{n\Delta t-}^{n\Delta t+} \sum_{n=\infty}^{\infty} f(t) \delta(t - n\Delta T) dt
\]

or

\[
f_n = f(n\Delta T)
\]

by the sifting property of \( \delta(t) \).

### 2.2 The Spectrum of the Sampled Waveform \( f^*(t) \):

Notice that sampling comb function \( s(t;\Delta T) \) is periodic and is therefore described by a Fourier series:

\[
s(t;\Delta T) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j\Omega_0 t}
\]

where all the Fourier coefficients are equal to \((1/\Delta T)\), and where \( \Omega_0 = 2\pi/\Delta T \) is the fundamental angular frequency. Using this form, the spectrum of the sampled waveform \( f^*(t) \) may be written

\[
F^*(j\Omega) = \int_{-\infty}^{\infty} f^*(t) e^{-j\Omega t} dt = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{j\Omega_0 t} e^{-j\Omega t} dt
\]

\[
= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(j(\Omega - n\Omega_0))
\]

\[
= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(j\left(\Omega - \frac{2\pi n}{\Delta T}\right))
\]

The Fourier transform of a sampled function \( f^*(t) \) is periodic in the frequency domain with period \( \Omega_0 = 2\pi/\Delta T \), and is a superposition of an infinite number of shifted replicas of the Fourier transform, \( F(j\Omega) \), of the original function scaled by a factor of \( 1/\Delta T \).