4.6 Laminar Boundary Layers

4.6.1 Assumptions

• 2D flow: $w, \frac{\partial}{\partial z} \equiv 0$ and $u(x,y), v(x,y), p(x,y), U(x,y)$.

• Steady flow: $\frac{\partial}{\partial t} \equiv 0$.

• For $\delta << L$, use local (body) coordinates $x, y$, with $x$ tangential to the body and $y$ normal to the body.

• $u \equiv$ tangential and $v \equiv$ normal to the body, viscous flow velocities (used inside the boundary layer).

• $U, V \equiv$ potential flow velocities (used outside the boundary layer).
4.6.2 Governing Equations

- Continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(1)

- Navier-Stokes:

\[
\begin{align*}
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\frac{u}{\partial x} + v \frac{\partial v}{\partial y} &= - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\end{align*}
\]  

(2)

(3)

4.6.3 Boundary Conditions

- KBC

Inside the boundary layer:

No-slip \( u(x, y = 0) = 0 \)

No-flux \( v(x, y = 0) = 0 \)

Outside the boundary layer the velocity has to match the P-Flow solution. Let \( y^* \equiv y/\delta \), \( x^* \equiv x/L \). Outside the boundary layer \( y^* \to \infty \) but \( y^* \to 0 \). We can write for the tangential and normal velocities

\[
\begin{align*}
u(x^*, y^* \to \infty) = U(x^*, y^* \to 0) &\Rightarrow u(x^*, y^* \to \infty) = U(x^*, 0), \\
v(x^*, y^* \to \infty) = V(x^*, y^* \to 0) &\Rightarrow v(x^*, y^* \to \infty) = V(x^*, 0) = 0
\end{align*}
\]

In short:

\[
\begin{align*}
u(x, y^* \to \infty) &= U(x, 0) \\
v(x, y^* \to \infty) &= 0
\end{align*}
\]

- DBC

As \( y^* \to \infty \), the pressure has to match the P-Flow solution. The \( x \)-momentum equation at \( y^* = 0 \) gives

\[
\begin{align*}
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 U}{\partial y^2} \\
\Rightarrow \frac{dp}{dx} &= -\rho U \frac{\partial U}{\partial x}
\end{align*}
\]
4.6.4 Boundary Layer Approximation

Assume that $R_{el} >> 1$, then $(u, v)$ is confined to a thin layer of thickness $\delta(x) << L$. For flows within this boundary layer, the appropriate order-of-magnitude scaling / normalization is:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Scale</th>
<th>Normalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$U$</td>
<td>$u = Uu^*$</td>
</tr>
<tr>
<td>$x$</td>
<td>$L$</td>
<td>$x = Lx^*$</td>
</tr>
<tr>
<td>$y$</td>
<td>$\delta$</td>
<td>$y = \delta y^*$</td>
</tr>
<tr>
<td>$v$</td>
<td>$V = ?$</td>
<td>$v = Vv^*$</td>
</tr>
</tbody>
</table>

- Non-dimensionalize the continuity, Equation (1), to relate $V$ to $U$

\[
\frac{U}{L} \left( \frac{\partial u}{\partial x} \right)^* + \frac{V}{\delta} \left( \frac{\partial v}{\partial y} \right)^* = 0 \implies V = O \left( \frac{\delta}{L} \frac{U}{U} \right)
\]

- Non-dimensionalize the $x$-momentum, Equation (2), to compare $\delta$ with $L$

\[
\frac{U^2}{L} \left( \frac{\partial u}{\partial x} \right)^* + \frac{U\nu}{\delta} \left( \frac{v}{\partial y} \right)^* = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\nu U}{\delta} \left[ \frac{\delta^2}{L^2} \left( \frac{\partial^2 u}{\partial x^2} \right)^* + \left( \frac{\partial^2 u}{\partial y^2} \right)^* \right]
\]

The inertial effects are of comparable magnitude to the viscous effects when:

\[
\frac{U^2}{L} \sim \frac{\nu U}{\delta^2} \implies \frac{\delta}{L} \sim \frac{\sqrt{\nu}}{UL} = \frac{1}{R_{el}} << 1
\]

The pressure gradient $\frac{\partial p}{\partial x}$ must be of comparable magnitude to the inertial effects

\[
\frac{\partial p}{\partial x} = O \left( \frac{\nu U^2}{L} \right)
\]
• Non-dimensionalize the $y$-momentum, Equation (3), to compare $\frac{\partial p}{\partial y}$ to $\frac{\partial p}{\partial x}$

\[
\frac{U \nu}{L} \left( \left( \frac{\partial v}{\partial x} \right)^* + \frac{\nu^2}{\delta} \left( \frac{\partial v}{\partial y} \right)^* \right) = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\nu V}{L^2} \left( \frac{\partial^2 v}{\partial x^2} \right)^* + \frac{\nu V}{\delta^2} \left( \frac{\partial^2 v}{\partial y^2} \right)^*
\]

The pressure gradient $\frac{\partial p}{\partial y}$ must be of comparable magnitude to the inertial effects

\[
\frac{\partial p}{\partial y} = O \left( \frac{U^2 \delta}{\rho L L} \right)
\]

Comparing the magnitude of $\frac{\partial p}{\partial x}$ to $\frac{\partial p}{\partial y}$ we observe

\[
\frac{\partial p}{\partial y} = O \left( \frac{U^2 \delta}{\rho L L} \right) \quad \text{while} \quad \frac{\partial p}{\partial x} = O \left( \frac{U^2}{\rho L} \right) \quad \Rightarrow
\]

\[
\frac{\partial p}{\partial y} << \frac{\partial p}{\partial x} \quad \Rightarrow \quad \frac{\partial p}{\partial y} \approx 0 \quad \Rightarrow
\]

\[
p = p(x)
\]

• Note:

- From continuity it was shown that $V/U \sim O(\delta/L) \Rightarrow v << u$, inside the boundary layer.

- It was shown that $\frac{\partial p}{\partial y} = 0$, $p = p(x)$ inside the boundary layer. This means that the pressure across the boundary layer is constant and equal to the pressure outside the boundary layer imposed by the external P-Flow.
4.6.5 Summary of Dimensional BVP

Governing equations for 2D, steady, laminar boundary layer

Continuity : \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\( x \)-momentum : \[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \]

\( y \)-momentum : \[ \frac{\partial p}{\partial y} = 0 \]

Boundary Conditions

\( KBC \)

At \( y=0 \) : \[ u(x,0) = 0 \]
\[ v(x,0) = 0 \]

At \( y/\delta \to \infty \) : \[ u(x,y/\delta \to \infty) = U(x,0) \]
\[ v(x,y/\delta \to \infty) = 0 \]

\( DBC \)

\[ \frac{dp}{dx} = -\rho U \frac{\partial U}{\partial x} \]

or \( p(x) = C - \frac{1}{2} \rho U^2(x,0) \)

Bernoulli for the P-Flow at \( y=0 \)

4.6.6 Definitions

Displacement thickness

\[ \delta^* \equiv \int_{0}^{\infty} \left( 1 - \frac{u}{U} \right) dy \]

Momentum thickness

\[ \theta \equiv \int_{0}^{\infty} \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy \]
**Physical Meaning of $\delta^*$ and $\theta$**

*Assume a 2D steady flow over a flat plate.*

Recall for steady flow over flat plate $\frac{\partial p}{\partial x} = 0$ and pressure $p = \text{const.}$

Choose a control volume ($[0, x] \times [0, y/\delta \to \infty]$) as shown in the figure below.

![Control Volume](https://via.placeholder.com/150)

CV for steady flow over a flat plate.

**Control Volume ‘book-keeping’**

<table>
<thead>
<tr>
<th>Surface</th>
<th>$\hat{n}$</th>
<th>$\vec{v}$</th>
<th>$\vec{v} \cdot \hat{n}$</th>
<th>$\vec{v}(\vec{v} \cdot \hat{n})$</th>
<th>$-p\hat{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\hat{i}$</td>
<td>$U_o\hat{i}$</td>
<td>$-U_o$</td>
<td>$-U_o^2\hat{i}$</td>
<td>$p\hat{i}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\hat{j}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$p\hat{j}$</td>
</tr>
<tr>
<td>3</td>
<td>$\hat{i}$</td>
<td>$u(x, y)\hat{i} + v(x, y)\hat{j}$</td>
<td>$u(x, y)$</td>
<td>$u^2(x, y)\hat{i} + u(x, y)v(x, y)\hat{j}$</td>
<td>$-p\hat{i}$</td>
</tr>
<tr>
<td>4</td>
<td>$\hat{j}$</td>
<td>$U_o\hat{i} + v(x, y)\hat{j}$</td>
<td>$v(x, y)$</td>
<td>$v(x, y)U_o\hat{i} + v^2(x, y)\hat{j}$</td>
<td>$-p\hat{j}$</td>
</tr>
</tbody>
</table>
Conservation of mass, for steady CV

\[
\oint_{1234} \vec{v} \cdot \hat{n} dS = 0 \Rightarrow -\int_0^{\infty} U_o dy' + \int_0^{\infty} u(x, y') dy' + \int_0^x v(x', y) dx' = 0 \Rightarrow \\
Q = \int_0^{\infty} U_o dy' - \int_0^{\infty} u dy' = \int_0^{\infty} (U_o - u) dy' = U_o \int_0^{\infty} \left(1 - \frac{u}{U_o}\right) dy' \Rightarrow Q = U_o \delta^* 
\]

where ()' are the dummy variables.

Conservation of momentum in x, for steady CV

\[
\oint_{1234} u(\vec{v} \cdot \hat{n}) dS = \sum F_x \Rightarrow \\
\int_0^{\infty} -U_o^2 dy' + \int_0^{\infty} u^2(x, y') dy' + \int_0^x v(x', y) U_o dx' = \int_0^{\infty} pdy' - \int_0^{\infty} pdy' + \sum F_{x,friction} \Rightarrow \\
\int_0^{\infty} -U_o^2 dy' + \int_0^{\infty} u^2(x, y') dy' + U_o \int_0^x v(x', y) dx' = \sum F_{x,friction} \Rightarrow \\
\int_0^{\infty} \left( -U_o^2 + u^2 + U_o^2 - U_o u \right) dy' = \sum F_{x,friction} \Rightarrow \\
U_o^2 \int_0^{\infty} \left( \frac{u^2}{U_o^2} - \frac{u}{U_o} \right) dy' = \sum F_{x,friction} \Rightarrow \\
\sum F_{x,friction} = -U_o^2 \int_0^{\infty} \left( 1 - \frac{u}{U_o} \right) dy' \Rightarrow \sum F_{x,friction} = -U_o^2 \theta 
\]
4.7 Steady Flow over a Flat Plate: Blasius’ Laminar Boundary Layer

Steady flow over a flat plate: BLBL

4.7.1 Derivation of BLBL

- **Assumptions** Steady, 2D flow. Flow over flat plate → \( U = U_0, V = 0, \frac{dp}{dx} = 0 \)
- **LBL governing equations**
  \[
  \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
  \]
  \[
  u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}
  \]
- **Boundary conditions**
  \( u = v = 0 \) on \( y = 0 \)
  \( v \to V = 0, u \to U_0 \) outside the BL, i.e., \( \left( \frac{y}{\delta} >> 1 \right) \)
- **Solution** Mathematical solution in terms of similarity parameters.
  \[
  \frac{u}{U} \quad \text{and} \quad \eta \equiv y \sqrt{\frac{U_o}{\nu x}} \Leftrightarrow \frac{y}{x} = \frac{\eta}{\sqrt{R_x}} \Leftrightarrow y = \eta \sqrt{\frac{\nu x}{U_o}}
  \]
  Similarity solution must have the form
  \[
  \frac{u(x, y)}{U_o} = F(\eta)
  \]
  self similar solution
We can obtain a PDE for $F$ by substituting into the governing equations. The PDE has no-known analytical solution. However, Blasius provided a numerical solution. Once again, once the velocity profile is evaluated we know everything about the flow.

4.7.2 Summary of BLBL Properties: $\delta, \delta_{0.99}, \delta^*, \theta, \tau_o, D, C_f$

\[
\begin{align*}
\frac{u(x,y)}{U_o} &= F(\eta); \quad \eta = y \sqrt{\frac{U_o}{\nu x}}; \quad y \equiv \eta \sqrt{\frac{\nu x}{U_o}}; \quad \frac{y}{x} = \frac{\eta}{\sqrt{R_x}} \\
\delta &\equiv \sqrt{\frac{\nu x}{U_o}} \\
\delta_{0.99} &\approx 4.9 \sqrt{\frac{\nu x}{U_o}}, \text{ i.e., } \eta_{0.99} = 4.9 \\
\delta^* &\approx 1.72 \sqrt{\frac{\nu x}{U_o}}, \text{ i.e., } \eta^* = 1.72 \\
\theta &\approx 0.664 \sqrt{\frac{\nu x}{U_o}} \\
\tau_o &\equiv \tau_w \approx 0.332 \rho U_o^2 \left(\frac{U_o x}{\nu}\right)^{-1/2} \\
&= 0.332 \left(\rho U_o^2\right) \underbrace{R_x^{-1/2}}_{\text{local } R^\#} \\
&= \frac{1}{\sqrt{x}} \\
&= U_o^{3/2}
\end{align*}
\]
Total drag on plate L x B

\[ D = B \int_{0}^{L} \tau_0 dx \cong 0.664 (\rho U_o^2) (BL) \left( \frac{U_o L}{\nu} \right)^{-1/2} \Rightarrow D \propto \sqrt{L}, \quad D \propto U^{3/2} \]

Friction (drag) coefficient:

\[ C_f = \frac{D}{\frac{1}{2} (\rho U_o^2) (BL)} \cong \frac{1.328}{\sqrt{Re_L}} \Rightarrow C_f \propto \frac{1}{\sqrt{L}}, \quad C_f \propto \frac{1}{\sqrt{U}} \]

Skin friction coefficient as a function of \( Re \).

**A look ahead: Turbulent Boundary Layers**

Observe form the previous figure that the function \( C_{f,\text{laminar}}(Re) \) for a laminar boundary layer is different from the function \( C_{f,\text{turbulent}}(Re) \) for a turbulent boundary layer for flow over a flat plate.

Turbulent boundary layers will be discussed in proceeding Lecture.
4.8 Laminar Boundary Layers for Flow Over a Body of General Geometry

The velocity profile given in BLBL is the exact velocity profile for a steady, laminar flow over a flat plate. What is the velocity profile for a flow over any arbitrary body? In general it is \( dp/dx \neq 0 \) and the boundary layer governing equations cannot be easily solved as was the case for the BLBL. In this paragraph we will describe a typical approximative procedure used to solve the problem of flow over a body of general geometry.

1. Solve P-Flow outside \( B \equiv B_0 \)
2. Solve boundary layer equations (with \( \nabla P \) term) \( \rightarrow \) get \( \delta^* (x) \)
3. From \( B_0 + \delta^* \rightarrow B \)
4. Repeat steps (1) to (3) until no change

- von Karman’s zeroth moment integral equation

\[
\frac{\tau_0}{\rho} = \frac{d}{dx} \left( U^2(x) \theta(x) \right) + \delta^*(x)U(x) \frac{dU}{dx}
\]  

(4)

- Approximate solution method due to Polthausen for general geometry \( (dp/dx \neq 0) \) using von Karman’s momentum integrals.

The basic idea is the following: we assume an approximate velocity profile (e.g. linear, 4th order polynomial, ...) in terms of an unknown parameter \( \delta(x) \). From the velocity profile we can immediately calculate \( \delta^*, \theta \) and \( \tau_o \) as functions of \( \delta(x) \) and the P-Flow velocity \( U(x) \).

Independently from the boundary layer approximation, we obtain the P-Flow solution outside the boundary layer \( U(x), \frac{dU}{dx} \).

Upon substitution of \( \delta^*, \theta, \tau_o, U(x), \frac{dU}{dx} \) in von Karman’s moment integral equation(s) we form an ODE for \( \delta \) in terms of \( x \).
Example for a 4th order polynomial Polthausen velocity profile

Polthausen profiles - a family of profiles as a function of a single parameter Λ(x) (shape function factor).

Assume an approximate velocity profile, say a 4th order polynomial:

\[ \frac{u(x,y)}{U(x,0)} = a(x) \left( \frac{y}{\delta} \right)^3 + b(x) \left( \frac{y}{\delta} \right)^2 + c(x) \left( \frac{y}{\delta} \right) + d(x) \left( \frac{y}{\delta} \right)^4 \]  

(5)

There can be no constant term in (5) for the no-slip BC to be satisfied \( y = 0 \), i.e., \( u(x,0) = 0 \).

We use three BC’s at \( y = \delta \)

\[ \frac{u}{U} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{at } y = \delta \]  

(6)

From (6) in (5), we re-write the coefficients \( a(x), b(x), c(x) \) and \( d(x) \) in terms of \( Λ(x) \)

\[ a = 2 + Λ/6, \quad b = -Λ/2, \quad c = -2 + Λ/2, \quad d = 1 - Λ/6 \]

To specify the approximate velocity profile \( \frac{u(x,y)}{U(x,0)} \) in terms of a single unknown parameter \( δ \) we use the x-momentum equation at \( y = 0 \), where \( u = v = 0 \)

\[ \frac{u}{\delta} \frac{\partial u}{\partial x} + \frac{v}{\delta} \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + v \frac{\partial^2 u}{\partial y^2} \bigg|_{y=0} \Rightarrow b = -\frac{1}{2} \left( \frac{dU}{dx} \frac{\delta^2}{\nu} \right) \Rightarrow Λ(x) = \frac{dU}{dx} \frac{\delta^2(x)}{\nu} \]

Observe: \( Λ \propto \frac{dU}{dx} \Rightarrow \begin{cases} Λ > 0 : & \text{favorable pressure gradient} \\ Λ < 0 : & \text{adverse pressure gradient} \end{cases} \)

Putting everything together:

\[ \frac{u(x,y)}{U(x,0)} = 2\left( \frac{y}{\delta} \right) - 2\left( \frac{y}{\delta} \right)^3 + \left( \frac{y}{\delta} \right)^4 + \frac{dU}{dx} \frac{\delta^2}{\nu} \left[ \frac{1}{6} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^2 + \frac{1}{2} \left( \frac{y}{\delta} \right)^3 - \frac{1}{6} \left( \frac{y}{\delta} \right)^4 \right] \]
\[ \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) \, dy = \delta \left(3 \frac{dU}{dx} \frac{\delta^2}{\nu} \right) \]

\[ \theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) \, dy = \delta \left(3 \frac{dU}{dx} \frac{\delta^2}{\nu} \right) - 1 \left(\frac{dU}{dx} \frac{\delta^2}{\nu}\right)^2 \]

\[ \tau_o = \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \frac{\mu U}{\delta} \left(2 + \frac{1}{6} \left(\frac{dU}{dx} \frac{\delta^2}{\nu}\right) \right) \]

Notes:
- Incipient flow \((\tau_o = 0)\) for \(\Lambda = -12\). However, recall that once the flow is separated the boundary layer theory is no longer valid.

- For \(\frac{dU}{dx} = 0 \rightarrow \Lambda = 0\) Pohlhausen profile differs from Blasius LBL only by a few percent.

After we solve the P-Flow and determine \(U(x), \frac{dU}{dx}\) we substitute everything into von Karman’s momentum integral equation (4) to obtain

\[ \frac{d\delta}{dx} = \frac{1}{U \frac{dU}{dx}} g(\delta) + \frac{d^2U}{dx^2} h(\delta) \]

where \(g, h\) are known rational polynomial functions of \(\delta\).

This is an ODE for \(\delta = \delta(x)\) where \(U, \frac{dU}{dx}, \frac{d^2U}{dx^2}\) are specified from the P-Flow solution.

General procedure:
1. Make a reasonable approximation in the form of (5),
2. Apply sufficient BC’s at \(y = \delta\), and the \(x\)-momentum at \(y = 0\) to reduce (5) as a function a single unknown \(\delta\),
3. Determine \(U(x)\) from P-Flow, and
4. Finally substitute into Von Karman’s equation to form an ODE for \(\delta(x)\). Solve either analytically or numerically to determine the boundary layer growth as a function of \(x\).