Flow inside a cylinder which is suddenly rotated

A Newtonian liquid with density $\rho$ and viscosity $\mu$ is initially at rest in a vertical, infinitely long cylinder of radius $R$. At time $0$, the cylinder starts to rotate with a constant rotational speed, $\Omega$ about its axis. This problem is a transient flow like the Rayleigh plate problem (or Stokes’ 1st problem). The governing equation is the following:

$$\frac{\partial V_\theta}{\partial t} = \nu \left( \frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial V_\theta}{\partial r} - \frac{V_\theta^2}{r^2} \right)$$  \hspace{1cm} (1)

The boundary/initial conditions are the following:

$$\begin{align*}
V_\theta &= \Omega R \quad \text{at} \quad r = R, \ 0 < t \\
V_\theta &= \text{finite} \quad \text{at} \quad r = 0, \ 0 \leq t \\
V_\theta &= 0 \quad \text{at} \quad t = 0, \ 0 \leq r \leq R
\end{align*}$$  \hspace{1cm} (2)

This problem can be solved using separation of variables. The solution finally leads to Bessel equation and there is a bit of algebra in the process. The final solution will be:

$$V_\theta(r, \theta) = \Omega r + 2\Omega R \sum_{k=1}^{\infty} \frac{J_1(\alpha_k r/R)}{\alpha_k J_0(\alpha_k)} \exp \left( -\frac{\nu \alpha_k^2 t}{R^2} \right)$$  \hspace{1cm} (3)

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in which $J_1$ and $J_0$ are respectively the first and zeroth order Bessel functions of the first kind and $\alpha_k$ is the $k$-$th$ root of $J_1$. 