2.58 HW5 Solutions

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Prob 10.1

According to eq. (10.19), for a harmonic oscillator,
\[ \Delta E = h \nu e \Delta \nu \]
\[ \Rightarrow \nu e = \frac{\Delta E}{h \nu e} = \frac{h \nu e}{h \nu e} = \frac{C_0}{\Delta \nu} \eta \]

From table 10.3, we have
\[ \nu e = C_0 \frac{\eta}{\Delta \nu} \approx \frac{3 \times 10^6}{1} \times 2143 = 6.429 \times 10^{13} \text{ Hz} \]
\[ \approx \frac{3 \times 10^6}{2} \times 4260 = 6.390 \times 10^{13} \text{ Hz} \]

Prob 10.7

(a) The Eiseeuser model.

The unit of the line strength (S) suggests that a mass absorption coefficient has been used.

At 500K and 1 atm,
\[ \rho = \rho_{sp} \cdot \frac{T_{sp}}{T} = 3 \times 10^{-3} \times \frac{273}{500} = 1.638 \times 10^{-3} \text{ g/cm}^3 \]
\[ X = \rho S = 1.638 \times 10^{-3} \text{ g/cm}^3 \times 50 \text{ cm} = 8.19 \times 10^{-2} \text{ g/cm}^2 \]
\[ \alpha = \frac{\beta X}{2 \pi \rho} = \frac{2.04 \times 10^{-4} \text{ cm}^{-1} \text{(g/cm)} \times 8.19 \times 10^{-2}}{2 \pi \times 0.04 \text{ cm}^{-1}} = 2.09 / \pi \]
\[ \beta = \frac{\pi \beta_{10}}{d} = \frac{\pi \times 0.04}{0.25} = 0.16 \pi \]
\[ I = 2 \beta \alpha = 0.669 \]

According to (10.38),
\[ L(x) = x [1 + (\frac{\pi x}{2})^{5/4}]^{-2/5} = 0.499 \]
\[ \bar{E}_1 = \text{erf} (\sqrt{\pi} \beta L(x)) = \text{erf} (\sqrt{\pi} \times 0.16 \pi \times 0.499) = 0.471 \]
(a) For simplicity, we will assume a constant average pressure of 0.5 atm for the atmosphere (see prob.10.20 for more accurate results). \( P_e = \left( \frac{P}{P_0} \right)^{n} = \left[ 0.5 (1 + 0.12 \frac{10^{-6}}{0.5}) \right]^{0.6} = 0.660 \) because

\[ X_1 = P_a \cdot L_1 = 1 \times 10^{-6} \text{ atm} \times 1 \times 10^5 \text{ cm} = 0.1 \text{ cm} \cdot \text{atm} \]

\[ \beta_1 = \frac{\gamma \cdot P_e}{0.145} = 0.145 \times 0.660 = 0.0957 \]

\[ \beta_2 = \frac{\gamma \cdot P_e}{0.377} = 0.377 \times 0.660 = 0.249 \]

\[ T_{01} = 2 \times X_1 / \omega_1 = \frac{2035 \text{ cm}^2 \text{ atm}^{-1} \times 0.1 \text{ cm} \cdot \text{atm}}{22 \text{ cm}^{-1}} = 9.25 \]

\[ A_1 = 2 \sqrt{T_{01} \beta_1 - \beta_1} = 2 \sqrt{9.25 \times 0.0957 - 0.0957} = 1.786 \]

\[ A_1 = A_1^* \omega_1 = 39.29 \text{ cm}^2 \]

\[ T_{02} = \frac{2 \times X_2}{\omega_2} = \frac{16 \times 1 \times 10^{-6}}{18.5} \]

\[ L_2 = 8.703 \text{ L_2} \times 1 \times 10^{-6} \]

By trial and error, we know \( 1/\beta < L_0 < \infty \)

\[ A_2^* = \ln (T_{02} \beta_2) + 2 - \beta = \ln (8.703 \times 10^{-6} \times 0.249 \times L_2) + 2 - 0.249 \]

\[ A_2 = A_2 \Rightarrow \]

\[ 39.29 = 18.5 \times [\ln (2.161 \times 10^{-6} \times L_2) + 1.751] \]

\[ \Rightarrow L_2 = 6.7 \times 10^5 \text{ cm} = 6.7 \text{ km} \]

(b) Assuming small change of \( L \), so that the correlations remain valid for \( A_1^* \) and \( A_2^* \).

All the empirical correlations listed in Table 10.2 are monotone increasing with \( T_0 \) (\( T_0 = \frac{\alpha \cdot P}{L} \)). By requiring \( A_1 = A_2 \), we know \( L_2 \) will decrease if \( L_1 \) was decreased. Also, a plot can show \( L_2/L_1 \) will decrease if \( L_1 \) ever changes.
Problem 10.29

According to Eq. (10.138),

\[ \varepsilon = \sum_n \left( \frac{E_{bn}}{T_3} \cdot \frac{\omega}{\sigma T} \right) \cdot A_n^* \]

Compare \( \frac{(\omega E_{bn})}{T_3} \) for all the bands, we find that only the 15 \( \mu \)m and 4.3 \( \mu \)m bands are important for CO\(_2\). The partial pressure of CO\(_2\) is given by:

\[
P_a = \frac{M P_y}{R_u T} = \frac{44.9/\text{mol} \times 0.25 \times 1.0 \times 10^5 \text{Pa}}{8.3144 \text{J/molK} \times 298} = 668.1 \text{ g/m}^3
\]

where \( y \) is the concentration percentage of CO\(_2\).

Use \( w_{0.01 \text{CO}_2} \) from Appendix F to yield

<table>
<thead>
<tr>
<th>Band (( \mu )m)</th>
<th>( y/y_0 )</th>
<th>( \phi/\phi_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.0</td>
<td>2.82133</td>
</tr>
<tr>
<td>4.3</td>
<td>1.0</td>
<td>2.44733</td>
</tr>
</tbody>
</table>

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Band (\( \mu \)m)} & \omega_1 & \omega_1 & \omega_1 \\
\hline
15               & 19.0       & 31.11      & 0.428       \\
4.3             & 110.0      & 27.43      & 1.481       \\
\hline
\end{array}
\]

(a) 0.01\% CO\(_2\)

\[
P_{e1} = \left[ \frac{P}{P_0} \left( 1 + (b-1) \frac{P_a}{\sigma T} \right) \right]^{1/\gamma} = \left[ \frac{0.01}{1} \left( 1 + 0.3 \times 1 \times 10^{-4} \right) \right]^{1/0.7} = 0.818
\]

\[
P_{e2} = 0.794
\]

\[
X_1 = X_2 = P_a L = 0.0668, \quad \beta_1 = \tau_1 P_{e1} = 0.350, \quad \beta_2 = 1.176
\]

\[
T_0 = \frac{\omega_1 X_1}{\omega_1} = 0.0408, \quad \tau_0 = 0.268
\]

\[
\Rightarrow A_1^* = 0.0408, \quad A_2^* = 0.268 \quad \text{(linear regime)}
\]

\[
\varepsilon = \left( \frac{E_{bn}}{T_3} \cdot \frac{\omega}{\sigma T} \right) A_1^* + \left( \frac{E_{bn}}{T_3} \cdot \frac{\omega_1}{\sigma T} \right) A_2^*
\]
\[ E = 1.3003 \times 10^{-8} \times \frac{31.11}{5.67 \times 10^{-8} \times 600} \times 0.0408 + 0.82748 \times 10^{-8} \times \frac{27.43 \times 0.268}{5.67 \times 10^{-8} \times 600} = 2.27 \times 10^{-3} \]

For (b) and (c), we can repeat the same procedures as in (a) to obtain the emissivity. The results are tabulated below.

<table>
<thead>
<tr>
<th>( \text{CO}_2 ) concentration</th>
<th>( E ) (wide band model)</th>
<th>( E ) (Leckner's model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01%</td>
<td>0.00227</td>
<td>0.00308</td>
</tr>
<tr>
<td>1%</td>
<td>0.0574</td>
<td>0.0431</td>
</tr>
<tr>
<td>100%</td>
<td>0.139</td>
<td>0.150</td>
</tr>
</tbody>
</table>