2.72 Elements of Mechanical Design
Spring 2009

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Elements of Mechanical Design

Lecture 12: Belt, friction, gear drives
Schedule and reading assignment

Quiz

- Bolted joint qualifying Thursday March 19th

Topics

- Belts
- Friction drives
- Gear kinematics

Reading assignment

- Read:
  14.1 – 14.7

- Skim:
  Rest of Ch. 14
Topic 1:
Belt Drives
Belt Drives

Why Belts?
- Torque/speed conversion
- Cheap, easy to design
- Easy maintenance
- Elasticity can provide damping, shock absorption

Keep in mind
- Speeds generally 2500-6500 ft/min
- Performance decreases with age

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Please see:
http://www.tejasthumpcycles.com/Parts/primaryclutch/3.35-inch-harley-Street-Belt-Drive.jpg
Belt Construction and Profiles

Many flavors
- Flat is cheapest, natural clutch
- Vee allows higher torques
- Synchronous for timing

Usually composite structure
- Rubber/synthetic surface for friction
- Steel cords for tensile strength
Belt Drive Geometry

\[ \omega_1 \quad d_1 \quad \text{Driving Pulley} \]

\[ \omega_2 \quad d_2 \quad \text{Driven Pulley} \]

Slack Side

Tight Side

\[ \text{v}_{\text{belt}} \]
Belt Drive Geometry

\[ \theta_1 \quad \text{span} \quad \theta_2 \]

\[ d_{\text{center}} \]
\[
\theta_1 = \pi - 2 \sin^{-1} \left( \frac{d_2 - d_1}{2d_{\text{center}}} \right) \\
\theta_2 = \pi + 2 \sin^{-1} \left( \frac{d_2 - d_1}{2d_{\text{center}}} \right)
\]
Belt Geometry

\[ d_{\text{span}} = \sqrt{d_{\text{center}}^2 - \left(\frac{d_2 - d_1}{2}\right)^2} \]

\[ L_{\text{belt}} = \sqrt{4d_{\text{center}}^2 - (d_2 - d_1)^2} + \frac{1}{2}(d_1 \theta_1 + d_2 \theta_2) \]
Drive Kinematics

\[ v_b = \frac{d_1}{2} \omega_1 = \frac{d_2}{2} \omega_2 \]

\[ \frac{d_1}{d_2} = \frac{\omega_2}{\omega_1} \]
Elastomechanics → torque transmission

- Kinematics → speed transmission
- Link belt preload to torque transmission

Proceeding analysis is for flat/round belt
Free Body Diagram

• Tensile force (F)
• Normal force (N)
• Friction force (μN)
• Centrifugal force (S)
Force Balance

Using small angle approx:

\[ \Sigma F_y = 0 = -(F + dF) \frac{d\theta}{2} - F \frac{d\theta}{2} + dN + dS \]

\[ Fd\theta = dN + dS \]

\[ \Sigma F_x = 0 = -\mu dN - F + (F + dF) \]

\[ \mu dN = dF \]
Let $m$ be belt mass/unit length

\[
dS = m \left( \frac{d}{2} \right)^2 \omega^2 d\theta
\]

Combining these red eqns:

\[
dF = \mu F d\theta - \mu m \left( \frac{d}{2} \right)^2 \omega^2 d\theta
\]

\[
\frac{dF}{d\theta} - \mu F = -\mu m \left( \frac{d}{2} \right)^2 \omega^2
\]
Belt Tension to Torque

Let the difference in tension between the loose side ($F_2$) and the tight side ($F_1$) be related to torque ($T$)

$$F_1 - F_2 = \frac{T}{d/2}$$

Solve the previous integral over contact angle and apply $F_1$ and $F_2$ as b.c.’s and then do a page of algebra:

$$F_{tension} = \frac{T}{d} \frac{e^{\mu \theta_{contact}} + 1}{e^{\mu \theta_{contact}} - 1}$$

$$F_1 = m \left( \frac{d}{2} \right)^2 \omega^2 + F_{tension} \frac{2e^{\mu \theta_{contact}}}{e^{\mu \theta_{contact}} + 1}$$

$$F_2 = m \left( \frac{d}{2} \right)^2 \omega^2 + F_{tension} \frac{2}{e^{\mu \theta_{contact}} + 1}$$

Used to find stresses in belt!!!
Practical Design Issues

Pulley/Sheave profile
- Which is right?

Manufacturer → lifetime eqs
- Belt Creep (loss of load capacity)
- Lifetime in cycles

Idler Pulley Design
- Catenary eqs → deflection to tension
- Large systems need more than 1

Images by v6stang on Flickr.
Practice problem

Delta 15-231 Drill Press

- 1725 RPM Motor (3/4 hp)
- 450 to 4700 RPM operation
- Assume 0.3 m shaft separation
- What is max torque at drill bit?
- What size belt?
- Roughly what tension?

Topic 2:
Friction Drives
Friction Drives

Why Friction Drives?
- Linear ↔ Rotary Motion
- Low backlash/deadband
- Can be nm-resolution

Keep in mind
- Preload → bearing selection
- Low stiffness and damping
- Needs to be clean
- Low drive force
Friction Drive Anatomy

Concerned with:

• Linear Resolution
• Output Force
• Max Roller Preload
• Axial Stiffness
Drive Kinematics/Force Output

Kinematics found from no slip cylinder on flat

\[ \Delta \delta_{bar} = \Delta \theta \cdot \frac{d_{wheel}}{2} \]

\[ v_{bar} = \omega_{wheel} \cdot \frac{d_{wheel}}{2} \]

Force output found from static analysis

- Either motor or friction limited

\[ F_{output} = \frac{2T_{wheel}}{d_{wheel}} \quad \text{where} \quad F_{output} \leq \mu F_{preload} \]
### Maximum Preload

**Variable Definitions**

\[ E_e = \left( \frac{1 - \nu_{wheel}^2}{E_{wheel}} + \frac{1 - \nu_{bar}^2}{E_{bar}} \right)^{-1} \]

\[ R_e = \left( \frac{1}{d_{wheel}^2} + \frac{1}{r_{crown}} \right)^{1/3} \]

\[ a_{contact} = \left( \frac{3F_{preload}R_e}{2E_e} \right)^{1/3} \]

**Shear Stress Equation**

\[ \tau_{wheel} = \frac{a_{contact}E_e}{2\pi R_e} \left( \frac{1 + 2\nu_{wheel}}{2} + \frac{2}{9} \cdot (1 + \nu_{wheel}) \cdot \sqrt{2(1 + \nu_{wheel})} \right) \]

**For metals:**

\[ \tau_{max} = \frac{3\sigma_y}{2} \]

\[ F_{preload, \ max} = \frac{16\pi^3 \tau_{max}^3 R_e^2}{3E_e^2 \left( \frac{1 + 2\nu_{wheel}}{2} + \frac{2}{9} \cdot (1 + \nu_{wheel}) \cdot \sqrt{2(1 + \nu_{wheel})} \right)^3} \]
Axial Stiffness

\[ k_{\text{axial}} = \left( \frac{1}{k_{\text{shaft}}} + \frac{1}{k_{\text{torsion}}} \frac{1}{d_{\text{wheel}}^2} + \frac{1}{k_{\text{tangential}}} + \frac{1}{k_{\text{bar}}} \right)^{-1} \]

\[ k_{\text{tangential}} = \frac{4a_e E_e}{(2-\nu)(1+\nu)} \]

\[ k_{\text{shaft}} = \frac{3\pi Ed_{\text{shaft}}^4}{4L^3} \]

\[ k_{\text{torsion}} = \frac{\pi Gd_{\text{wheel}}^4}{32L} \]

\[ k_{\text{bar}} = \frac{EA_{c,\text{bar}}}{L} \]
Friction Drives

Proper Design leads to
- Pure radial bearing loads
- Axial drive bar motion only

Drive performance linked to motor/transmission
- Torque ripple
- Angular resolution

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Topic 3: Gear Kinematics
why gears?

- Torque/speed conversion
- Can transfer large torques
- Can run at low speeds
- Large reductions in small package

keep in mind

- Requires careful design
- Attention to tooth loads, profile
Gear Types and Purposes

Spur Gears
- Parallel shafts
- Simple shape → easy design, low $$$
- Tooth shape errors → noise
- No thrust loads from tooth engagement

Helical Gears
- Gradual tooth engagement → low noise
- Shafts may or may not be parallel
- Thrust loads from teeth reaction forces
- Tooth-tooth contact pushes gears apart

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Gear Types and Purposes

**Bevel Gears**
- Connect two intersecting shafts
- Straight or helical teeth

**Worm Gears**
- Low transmission ratios
- Pinion is typically input (Why?)
- Teeth sliding $\rightarrow$ high friction losses

**Rack and Pinion**
- Rotary $\leftrightarrow$ Linear motion
- Helical or straight rack teeth

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Tooth Profile Impacts Kinematics

Want constant speed output

- Conjugate action = constant angular velocity ratio
- Key to conjugate action is to use an involute tooth profile

Output speed of gear train

\[ \omega_{out} \text{ [rpm]} \]

Time [sec]

“Ideal” involute/gear

“Real” involute/gear

Non or poor involute

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Instantaneous Velocity and Pitch

Model as rolling cylinders (no slip condition):

\[ \vec{v} = \vec{\omega}_1 \times \vec{r}_1 = \vec{\omega}_2 \times \vec{r}_2 \]

\[ \frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} \]

Model gears as two pitch circles

- Contact at pitch point

Pitch Circles Meet @ Pitch Pt.
Instantaneous Velocity and Pitch

Meshing gears must have same pitch

- $N_g$ = # of teeth, $D_p$ = Pitch circle diameter

Diametral pitch, $P_D$:

Circular pitch, $P_C$:

$$P_D = \frac{N_g}{D_p}$$

$$P_C = \frac{\pi D_p}{N_g} = \frac{\pi}{P_D}$$
Drawing the Involute Profile

• Gear is specified by diametral pitch and pressure angle, $\Phi$

\[ D_B = D_P \cos \Phi \]
Drawing the Involute Profile

\[ L_n = n \frac{D_B}{2} \Delta \theta \]
Transmission Ratio for Serial Gears

Transmission ratio for elements in series:

\[ TR = \left( \text{proper sign} \right) \cdot \frac{\omega_{\text{out}}}{\omega_{\text{in}}} \]

From pitch equation:

\[ \frac{P_1}{D_1} = \frac{N_1}{D_2} = P_2 \]

For Large Serial Drive Trains:

\[ TR = \left( \text{proper sign} \right) \cdot \frac{\text{Product of driving teeth}}{\text{Product of driven teeth}} \]
Transmission Ratio for Serial Gears

Serial trains:

\[ TR = \text{(proper sign)} \cdot \frac{\text{Product of driving teeth}}{\text{Product of driven teeth}} \]

Example 1:

\[ TR = ? \]

Example 2:

\[ TR = ? \]
Example 3: Integral gears in serial gear trains

What is TR? Gear 1 = input and 5 = output

\[
TR = (\text{proper sign}) \cdot \frac{\text{Product of driving teeth}}{\text{Product of driven teeth}}
\]

Gear - 1
\( N_1 = 9 \)

Gear - 2
\( N_2 = 38 \)

Gear - 3
\( N_3 = 9 \)

Gear - 4
\( N_4 = 67 \)

Gear - 5
\( N_5 = 33 \)
Planetary gear trains are very common

- Very small/large TRs in a compact mechanism

Terminology:
How do we find the transmission ratio?

Image removed due to copyright restrictions. Please see http://www.cydgears.com.cn/products/Planetarygeartrain/planetarygeartrain.jpg
Planetary Gear Train TR

If we make the arm stationary, than this is a serial gear train:

\[
\frac{\omega_{ra}}{\omega_{sa}} = \frac{\omega_{ring} - \omega_{arm}}{\omega_{sun} - \omega_{arm}} = TR
\]

\[
TR = -\frac{N_{sun}}{N_{planet}} \cdot \frac{N_{planet}}{N_{ring}} = -\frac{N_{sun}}{N_{ring}}
\]

\[
\frac{\omega_{pa}}{\omega_{sa}} = \frac{\omega_{planet} - \omega_{arm}}{\omega_{sun} - \omega_{arm}} = TR
\]

\[
TR = -\frac{N_{sun}}{N_{planet}}
\]
Planetary Gear Train Example

If the sun gear is the input, and the ring gear is held fixed:

\[
\omega_{ra} = \frac{0 - \omega_{arm}}{\omega_{sun} - \omega_{arm}} = TR
\]

\[
TR = -\frac{N_{sun}}{N_{planet}} \cdot \frac{N_{planet}}{N_{ring}} = -\frac{N_{sun}}{N_{ring}}
\]

\[
\omega_{output} = \omega_{arm} = \frac{TR}{TR - 1} \omega_{sun}
\]
Case Study: Cordless Screwdriver

Given: Shaft $T_{SH}(\omega_{SH})$ find motor $T_M(\omega_{SH})$

- Geometry dominates relative speed (Relationship due to TR)

2 Unknowns: $T_M$ and $\omega_M$ with 2 Equations:
- Transmission ratio links input and output speeds
- Energy balance links speeds and torques
Example: DC Motor shaft

\[ T(\omega) = T_S \cdot \left( 1 - \frac{\omega}{\omega_{NL}} \right) \]

\( P(\omega) \) obtained from \( P(\omega) = T(\omega) \cdot \omega \)

Speed at maximum power output:

\[ P(\omega) = T(\omega) \cdot \omega = T_S \cdot \left( \omega - \frac{\omega^2}{\omega_{NL}} \right) \]

\[ \omega_{PMAX} = \frac{\omega_{NL}}{2} \]

\[ P_{MAX} = T_S \cdot \left( \frac{\omega_{NL}}{4} \right) \]
Example: Screw driver shaft

A = Motor shaft torque-speed curve
What is the torque-speed curve for the screw driver?

Train ratio = $1/81$

MOTOR SHAFT
$T_M$, $\omega_M$

GEAR train # 1

Electric Motor

GEAR train # 2

SCREW DRIVER SHAFT
$T_{SH}$, $\omega_{SH}$

System boundary
Example: Screw driver shaft

C = Motor shaft power curve

What is the power-speed curve for the screw driver?

Train ratio = $\frac{1}{81}$