Lecture # 2, Quantum Computation 2: QEC Criteria

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Outline:

0. Review

1. Classical Coding

2. Q. Coding

3. Operator Measurement and Error Syndromes

4. Shor 9 Qubit Code

5. Quantum Error correction Codes Criteria (QEC criteria)
0. Review

\[ \rho \xrightarrow{\varepsilon} \rho' \]

\[ \varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger \text{ where } \sum_k E_k E_k^\dagger = I \]
1. CLASSICAL CODING

![Diagram of a binary symmetric channel]

FIG. 1: a binary symmetric channel

\[ P = \text{prob of error} \]

Definition: A Classical \([n,k,d]\) code is a set of \(2^k\) n-bit strings which have a minimum Hamming distance \(d\).

Definition: A Hamming distance between two bit strings is \(d(x, y) = w(x \oplus y)\) where \(\oplus\) is the x-or operator, and \(w\) is an operation that counts the number of ones.

Example: \(0_{L(ogical)} = 000\), \(1_L = 111\) is a \([3,1,3]\) code

<table>
<thead>
<tr>
<th>could send</th>
<th>could receive</th>
<th>prob</th>
<th>decode</th>
<th>prob. of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0_L = 000</td>
<td>000</td>
<td>((1 - p)^3)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>001</td>
<td>(p(1 - p)^2)</td>
<td>0</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>010</td>
<td>(p(1 - p)^2)</td>
<td>0</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>(p(1 - p)^2)</td>
<td>0</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>011</td>
<td>(p^2(1 - p))</td>
<td>1</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>101</td>
<td>(p^2(1 - p))</td>
<td>1</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>(p^2(1 - p))</td>
<td>1</td>
<td>(p^2(1 - p))</td>
</tr>
<tr>
<td></td>
<td>111</td>
<td>(p^3)</td>
<td>1</td>
<td>(p^3)</td>
</tr>
</tbody>
</table>

So, the total probability of error is \(3p^2 - 2p^3 = O(p^2)\)
2. QUANTUM CODING

1995: Thought error correction to be impossible!

1. States collapse on measurement

2. Classically error occurs or does not occur. In Q. M., errors are continuous: $\alpha|0\rangle + \beta|1\rangle \to (\alpha + \varepsilon)|0\rangle + ...$

3. No cloning Thm prohibits copying, so cannot create $\alpha|0\rangle + \beta|1\rangle \to (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$

The Solutions:

1. Measure only the effect of the environment, not the state (i.e. did an error occur?)

2. & 3. Orthogonalize errors using entanglement: the environment has done one thing, or another, in an entangled way. $\alpha|didsomething\rangle + \beta|didthing\rangle$

Example: The Quantum Bit Flip Code:

$|0_L\rangle = |000\rangle$

$|1_L\rangle = |111\rangle$

$|\Phi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$

suppose $\varepsilon(\rho) = (1 - P)\rho + PX\rho X$ where $P$ is the probability of error and $X$ is the error operator.

$A \xrightarrow{\varepsilon(\rho)} B$

Define: An $[[n,k]]$ quantum code $C$ is a k-qubit subspace of an n-qubit Hilbert space. So, for our example, $k=3$, $n=1$. 
### 3. OPERATOR MEASUREMENT

Given $U$ with eigenvalues $\pm 1$, eigenvectors $|u_{\pm}\rangle$

Definition: Measuring $U$

$$ |0\rangle \quad \text{H} \quad \text{H} \quad \text{X} \quad \text{measurement result} \quad z $$

Initially, the state is $|0\rangle(C_0|u_+\rangle + C_1|u_-\rangle)$

After the first Hadamard, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(C_0|u_+\rangle + C_1|u_-\rangle)$

After the Controlled-U gate, $\frac{1}{\sqrt{2}}(|0\rangle(C_0|u_+\rangle + C_1|u_-\rangle) + |1\rangle(C_0|u_+\rangle - C_1|u_-\rangle))$

After the last Hadamard, $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(C_0|u_+\rangle + C_1|u_-\rangle) + (|0\rangle - |1\rangle)(C_0|u_+\rangle - C_1|u_-\rangle)$

$= |0\rangle C_0|u_+\rangle + |1\rangle C_1|u_-\rangle$

If the measurement is $z = 0$, then $|Psi\rangle = |u_+\rangle$. (With prob $C_0^2$, $z = 0$)

If the measurement is $z = 1$, then $|Psi\rangle = |u_-\rangle$. (With prob $C_1^2$, $z = 1$)

#### 3.1. Error Correction Syndrome Measurement

$$ U_1 = \sigma_z^1 \sigma_z^2 = \sigma_z \sigma_z I $$

$$ U_2 = \sigma_z^2 \sigma_z^3 = I \sigma_z \sigma_z $$
\[\begin{array}{|c|c|c|}
\hline
\text{state} & U_1 & U_2 \\
\hline
\alpha|000\rangle + \beta|111\rangle & 0 & 0 \\
\alpha|001\rangle + \beta|110\rangle & 0 & 1 \\
\alpha|010\rangle + \beta|101\rangle & 1 & 1 \\
\alpha|100\rangle + \beta|011\rangle & 1 & 0 \\
\hline
\end{array}\]

**TABLE I:** 0 represents a +1 eigenstate of \(U_i\), and 1 represents a -1 eigenstate.

Steps to Error Correction:

1. measure syndrome operators (here, \(U_1\) & \(U_2\))

2. Apply recovery operator \(R\) (here, 00 \(\equiv\) \(I\), 01 \(\equiv\) \(\delta_3\), 11 \(\equiv\) \(\delta_2\), 10 \(\equiv\) \(\delta_1\))

To create the initial state \(|\Psi_{si_L}\rangle\):

\[
\alpha|0\rangle + \beta|1\rangle \quad \alpha|000\rangle + \beta|111\rangle
\]

And then to error correct:

\[
\begin{array}{c}
|\psi\rangle \\
\hline
E \\
\hline
|\psi\rangle \quad H \quad H \quad \times \\
\hline
|\psi\rangle \quad H \quad H \quad \times \\
\hline
\end{array}
\]

\[
R
\]

note: the double lines indicate classical information.
Claim:
This scheme also corrects for a small continuous rotation error!

We will do this on one bit to demonstrate.
\[
\varepsilon(\rho) = e^{i \sigma_x \rho} e^{i \sigma_x}
\]
\[
e^{-i \sigma_x} = R_x(2\epsilon)
\]
\[
R_x(2\epsilon)|\Psi\rangle \cong |\Psi\rangle - i\epsilon \sigma_x^1 |PSi\rangle \equiv |\Psi'\rangle
\]
The fidelity is \( F = \sqrt{\langle \Psi | \Psi' \rangle^2} \equiv 1 - \epsilon \)

Syndrome measurement collapses error into either I or \( \sigma_x^1 \)
\[
F(R(\varepsilon(\rho)), |\Psi\rangle) \equiv ? \equiv 1 - \epsilon^2
\]

Example: The Phase Flip Code
\[
\varepsilon_{\text{phase} \text{flip}}(\rho) = (1 - P)\rho + P\sigma_z \rho \sigma_z
\]
Recall \( H\sigma_z H = \sigma_z, H\sigma_x H = \sigma_x \)
So, \( H\varepsilon_{\text{phase} \text{flip}}(H\rho H) = \varepsilon_{\text{bit} \text{flip}} \)

Explicitly,

![Diagram of quantum circuit](image)

For the bit flip: \( U_o = \sigma_z \sigma_z I \) and \( U_1 = I \sigma_z \sigma_z \)

For the phase flip: \( U_o = \sigma_x \sigma_x I \) and \( U_1 = I \sigma_x \sigma_x \)

Claim:

arbitrary errors can be described as \( \sigma_x, \sigma_z \), and \( \sigma_x \sigma_z \) errors

Proof Argument:
\[
\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger
\]
where we are guaranteed \( \sum_k E_k E_k^\dagger = I \)

Recall pauli matrices \( \sigma_j = I, \sigma_x, \sigma_y, \sigma_z \), and that \( \sigma_y = -i \sigma_x \sigma_z \)

Since \( \sigma_j \) is a basis for all 2x2 hermitian matrices, let \( E_k = \sum_j C_{kj} \sigma_j \).
Then, $\varepsilon(\rho) = \sum_{k, j, j'} C_k C_{k'}^{*} \sigma_j \rho \sigma_{j'}$

$\varepsilon(\rho) \sum_{j, j'} \chi_{jj'} \sigma_j \rho \sigma_{j'}$ is the “Chi representation or OSR”

Example: recall

$R_x(2\epsilon) |\Psi\rangle \cong |Psi\rangle - i\epsilon \sigma_x |Psi\rangle$

$\varepsilon(\rho) = \rho - i\epsilon \sigma_x \rho - i\epsilon \rho \sigma_x + \epsilon^2 \sigma_x \rho \sigma_x$

The $-i\epsilon \sigma_x \rho - i\epsilon \rho \sigma_x$ term disappears in the syndrome measurement, and the $\rho + \epsilon^2 \sigma_x \rho \sigma_x$ term remains.

The result is that the syndrome measurement projects the environment into a definite error state.
4. **SHOR 9 QUBIT CODE**

\[
|0_L\rangle = (|000\rangle + |111\rangle) \otimes 3 / \sqrt{8}
\]
\[
|1_L\rangle = (|000\rangle - |111\rangle) \otimes 3 / \sqrt{8}
\]

this code will correct ANY single qubit error.

Syndrome Measurements:

for a bit flip: \(\sigma_x^1 \sigma_z^2, \sigma_x^2 \sigma_z^3, \sigma_x^4 \sigma_z^5, \sigma_x^5 \sigma_z^6, \sigma_x^7 \sigma_z^8, \sigma_x^8 \sigma_z^9\)

for a phase flip: \(\sigma_x^1 \sigma_z^2, \sigma_x^3 \sigma_z^4, \sigma_x^4 \sigma_z^5, \sigma_x^5 \sigma_z^6, \sigma_x^6 \sigma_z^7, \sigma_x^8 \sigma_z^9\)
5. QEC CRITERIA/CONDITIONS

Channel: \( E(\rho) = \sum_k E_k \rho E_k^\dagger \)

Thm: Let \( C \) be a quantum Code defined by the orthonormal states \( \{ |\Psi_i\rangle \} \)
\( \exists \) a quantum recovery operation \( R \) correction \( \epsilon \) on \( C \) iff:

1. Orthogonality:
   
   if I have 2 errors \( j \) and \( k \),
   
   \[ \langle \Psi_l | E_j^\dagger E_k | \Psi_l \rangle = 0 \]

2. Nondeformation criteria:

   \[ \langle \Psi_l | E_k^\dagger E_k | \Psi_l \rangle = d_k \forall l \]
   
   this is so you cannot distinguish shrinking on different code words, all shrinking is the same.

\[ \sum_k E_k^\dagger E_k = 1 \]

Proof: \((\rightarrow)\)

Let \( P = \sum_l |\Psi_i\rangle \langle \Psi_l| \) (project onto \( C \))

note \( PE_j^\dagger E_k P = d_k \delta_{jk} P \) \((\rightarrow)\)

note by Polar decomposition (extracting rotation and shrinkage) \( E_k P = U_k \sqrt{PE_k^\dagger E_k P} = \sqrt{d_k} U_k P \) where \( \sqrt{d_k} \) is the shrinkage and \( U_k P \) is the rotation.

1. Syndrome measurement:
let $P_k = U_k P U_k^\dagger = \frac{E_k P U_k^\dagger}{\sqrt{d_k}} = \frac{u_k P E_j^\dagger}{\sqrt{d_k}}$

By (*), the $P_k$s are orthogonal:

\[ \forall k \neq j, \quad P_k P_j \propto U_k P E_k^\dagger E_j P U_j^\dagger = 0 \]

measure $P_k$ output $k$ syndrome.

2. Apply Recovery $R$

\[ R(\rho) = \sum_k U_k^\dagger P_k \rho P_k U_k \]

note for $|\Psi\rangle \in C$,

\[ U_k^\dagger P_k E_j |\Psi\rangle = \frac{U_k^\dagger U_k P E_j^\dagger}{\sqrt{d_k}} E_j P |\Psi\rangle = \frac{\delta_{jk} d_j P}{\sqrt{d_k}} |\Psi\rangle = \sqrt{d_k} \delta_{jk} |\Psi\rangle \]

Thus:

\[ R(\varepsilon(|\Psi\rangle \langle \Psi|)) = R(\sum_j E_j |\Psi\rangle \langle \Psi| E_j^T) = \sum_{jk} U_k^\dagger P_k E_j E_j^\dagger P_k U_k = \sum_{jk} d_k \delta_{jk} P = |\Psi\rangle \langle \Psi| \]