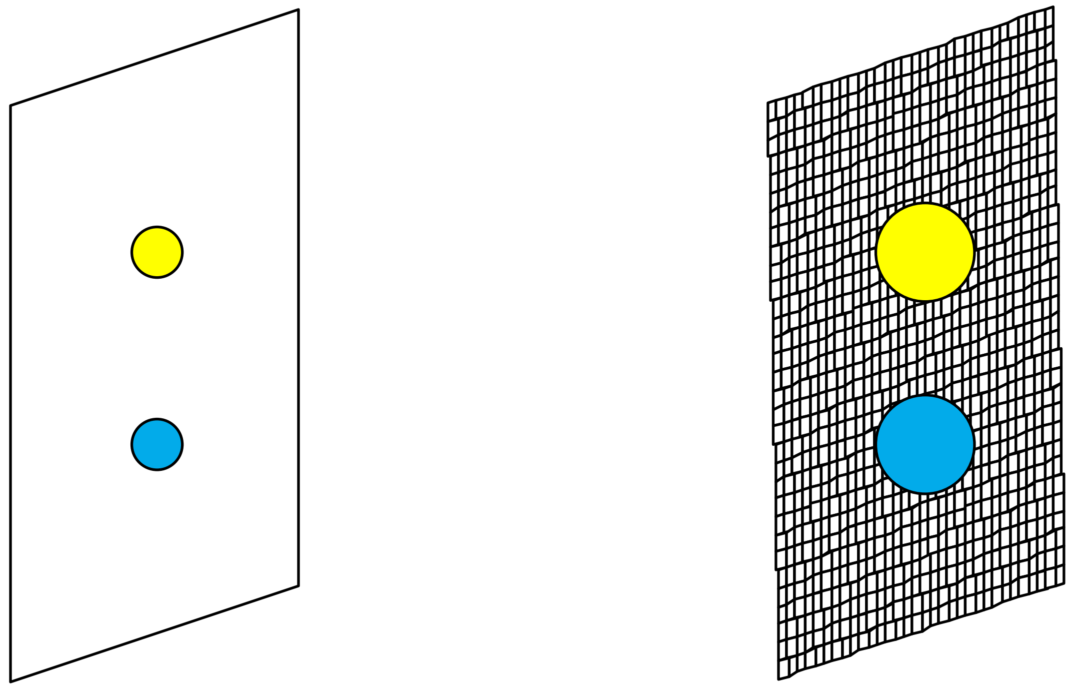
# Sampling

## Outline

- FT of comb function
- Sampling
- Nyquist Condition
- sinc interpolation
- Truncation
- Aliasing

# Sampling



As we saw with the CCD camera and the pinhole imager, the detector plane is not a continuous mapping, but a discrete set of sampled points. This of course limits the resolution that can be observed.

# Limits of Sampling

- Finite # of data points
- Finite field of view
- High spatial frequency features can be missed or recorded incorrectly

# Formalism of Sampling

$$\{f_n\} = \int_{-\infty}^{\infty} TopHat\left(\frac{2x}{fov}\right) Comb\left(\frac{x}{\Delta x}\right) f(x)\,dx$$

recall that

$$Comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)\ \ dx$$

$$TopHat(x) = \begin{cases} 1; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$$

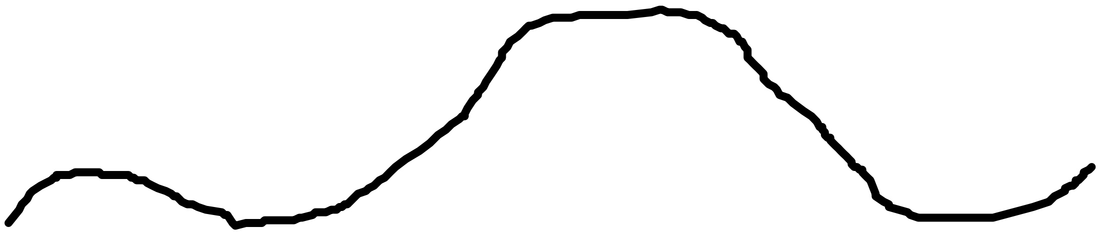
# Formalism of Sampling

$$\{f_n\} = \int_{-\infty}^{\infty} TopHat\left(\frac{2x}{fov}\right) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) f(x) dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{\frac{-fov}{2}}^{\frac{+fov}{2}} f(x)\delta(x - n\Delta x)$$

$$f_n = \int_{\frac{-fov}{2}}^{\frac{+fov}{2}} f(x)\delta(x - n\Delta x) dx$$

# Formalism of Sampling

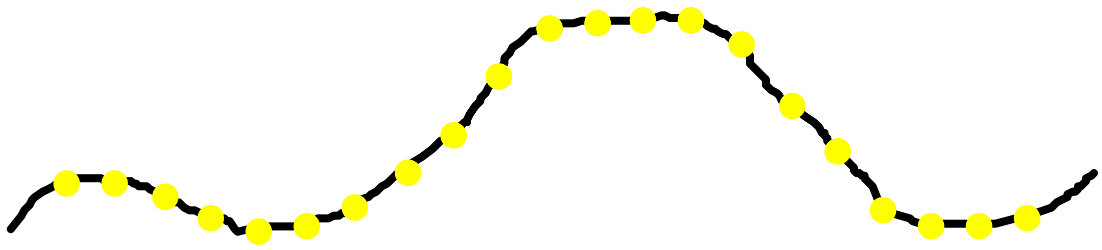


**f(x)**

**Comb(x/Δx)**

**TopHat(2x/fov)**

# Formalism of Sampling



**f(x)**

**Comb(x/Δx)**

**TopHat(2x/fov)**

# Formalism of Sampling

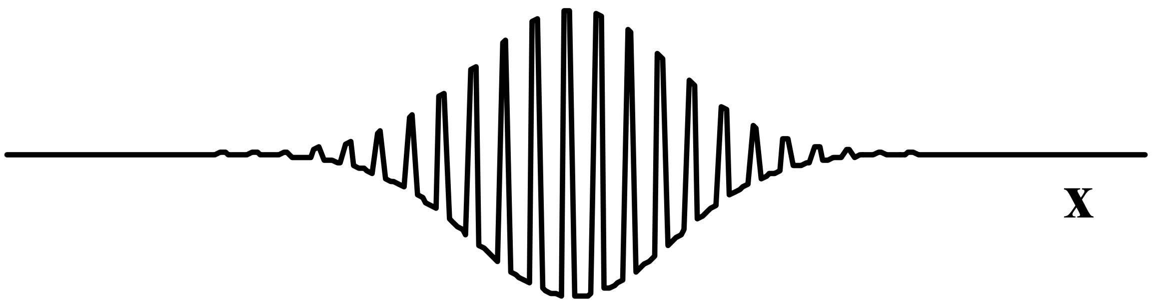


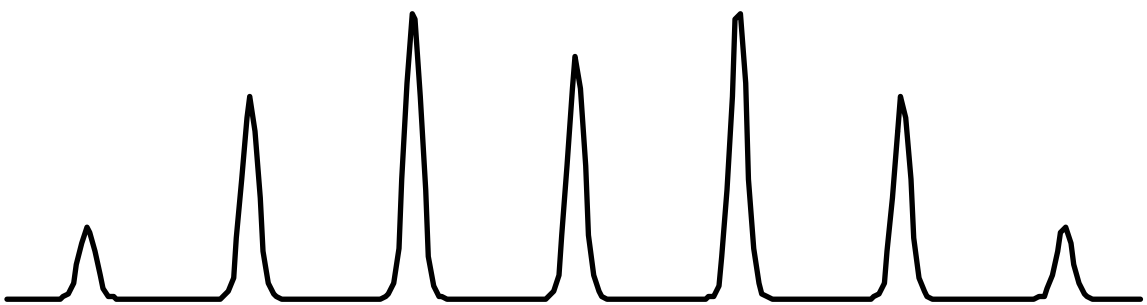
$\{f_n\}$

# Frequency of Sampling

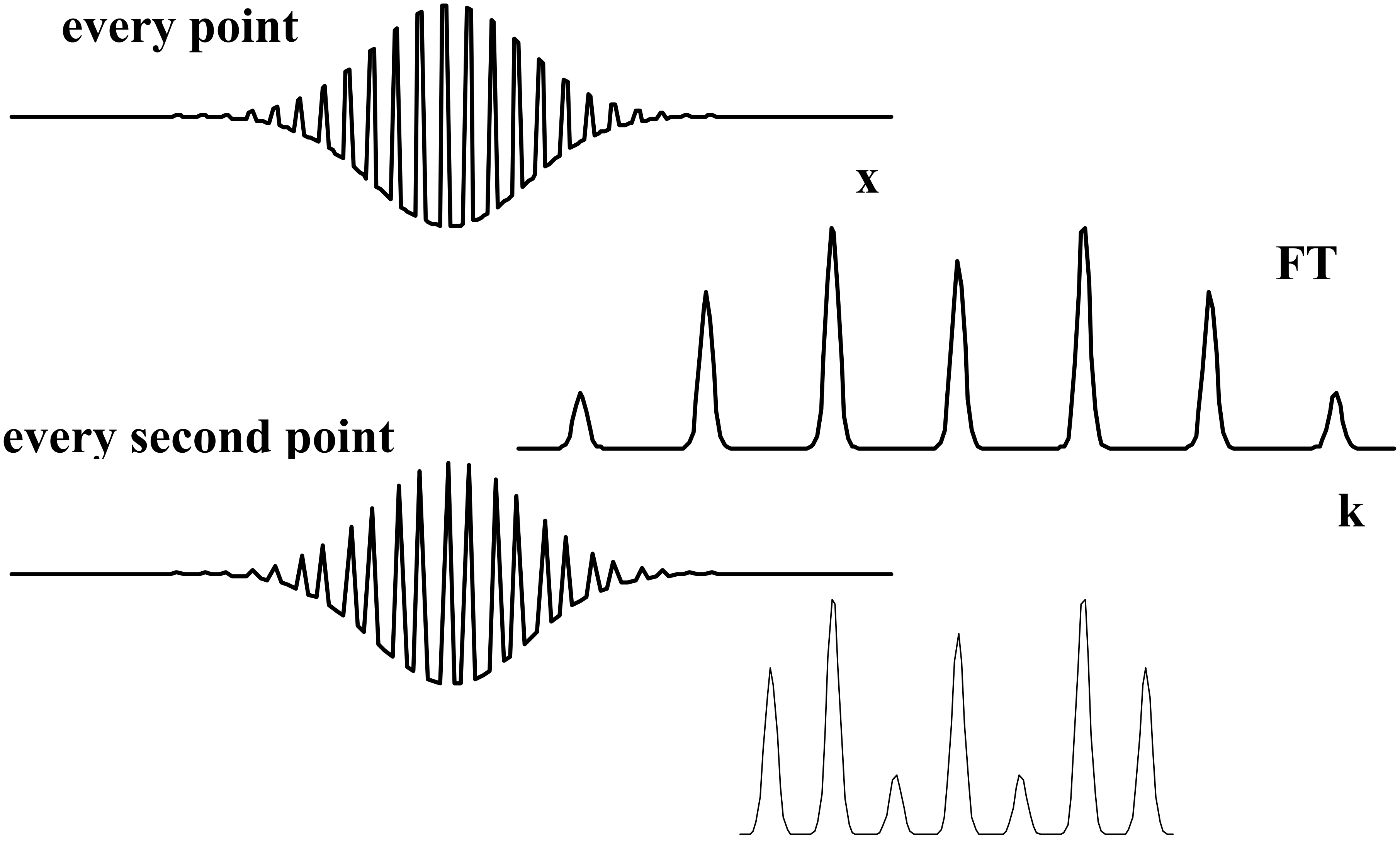
```
data = Table @If @Mod@x , 7D < 2, 1, - 1D * Exp @- Hx - 128 L^ 2 ê 1000 D, 8x , 1, 256 <D;

ListPlot @data , 8PlotJoined -> True , PlotRange -> All , Axes -> False , AspectRatio -> 1 ê 4,
  PlotStyle -> Thickness @0.005 D<D
```

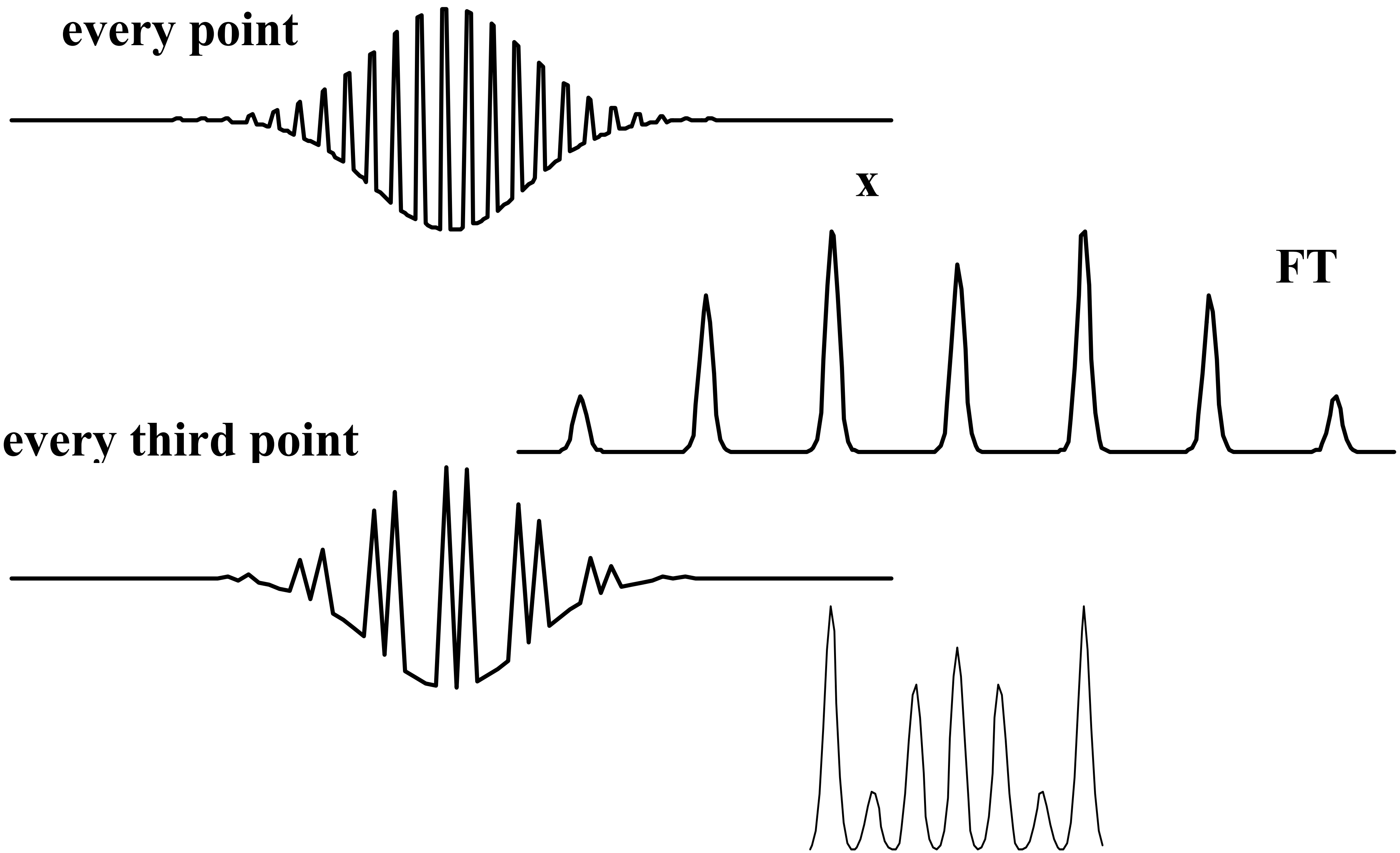


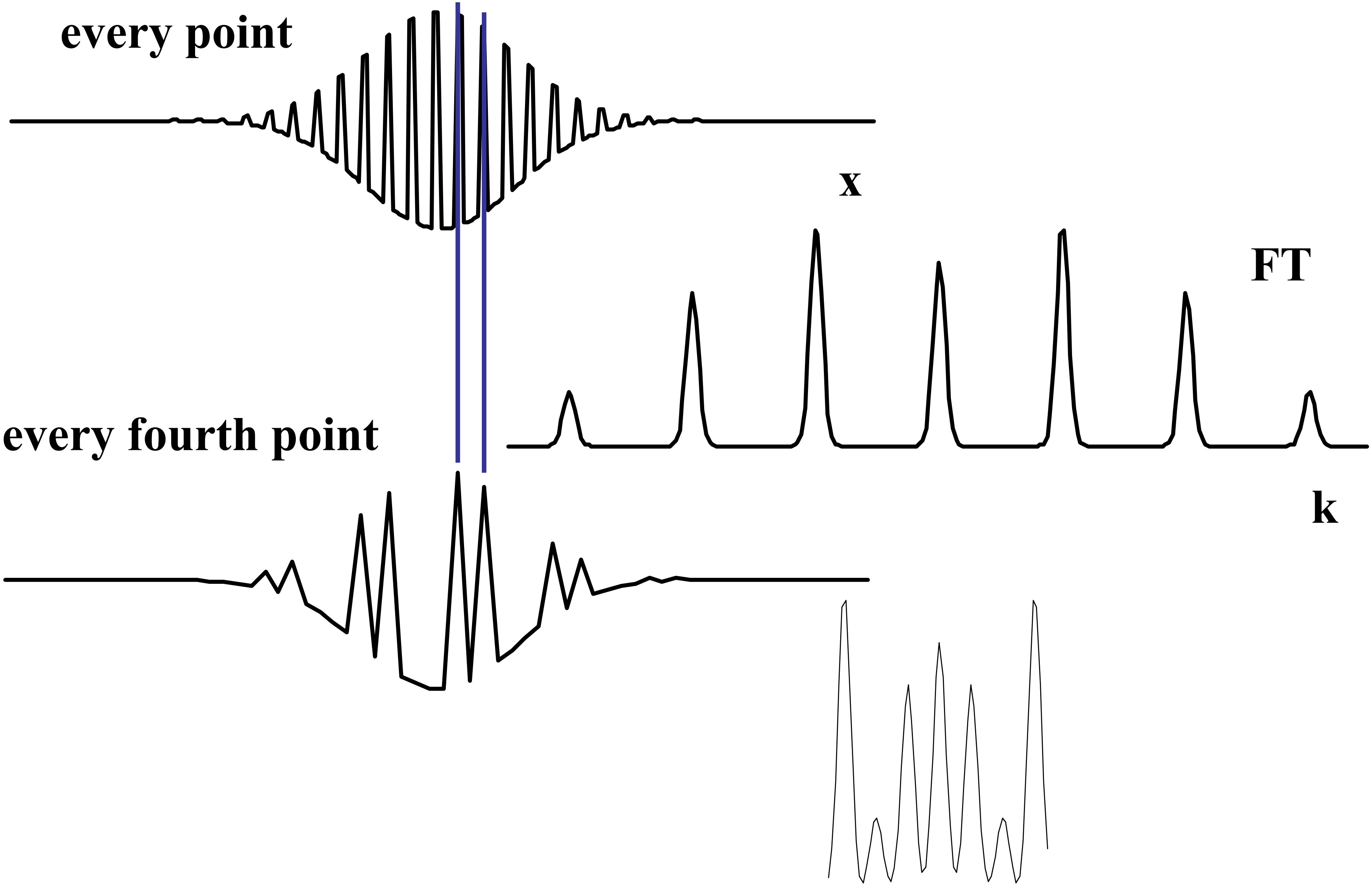
# Frequency of Sampling



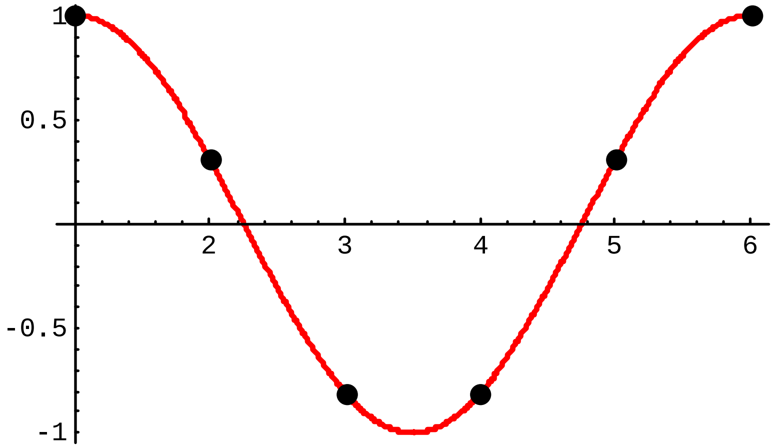
# Frequency of Sampling



# Frequency of Sampling



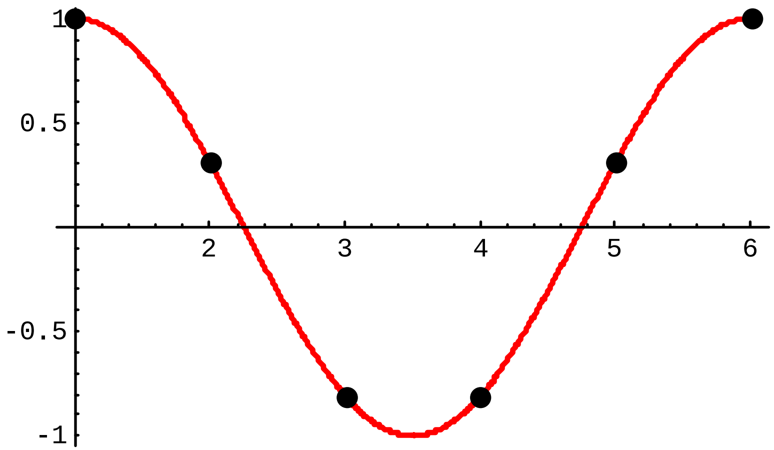
# Sampling

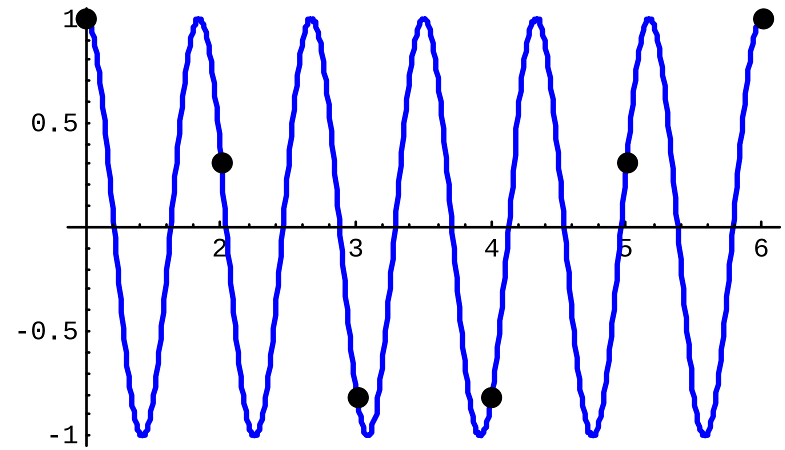



```
In[19]:= p2 = Plot @Cos @2 Pi 0.2 Hx - 1 LD, 8x, 1, 6<,
    8PlotPoints -> 512 ,
     PlotStyle -> 8Thickness @0.01 D, RGBColor @1 , 0, 0 D<<D
```

```
In[11]:= p4 = ListPlot @Table @Cos @2 Pi 1.2 x D, 8x, 0, 5, 1 <D,
    Prolog -> AbsolutePointSize @10 DD
```

# Sampling



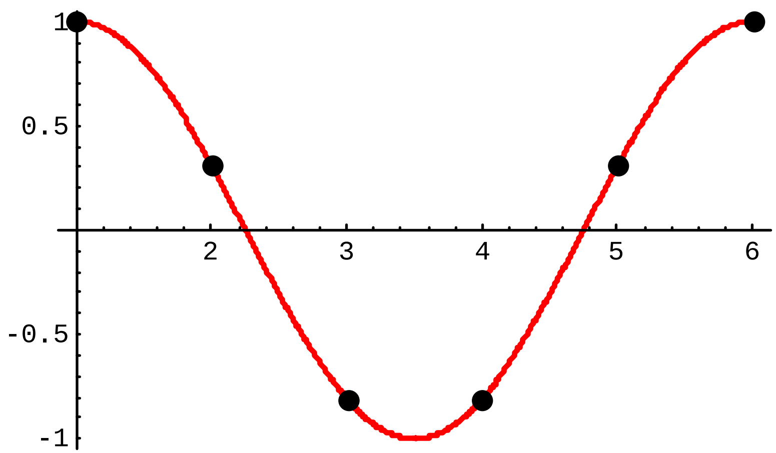



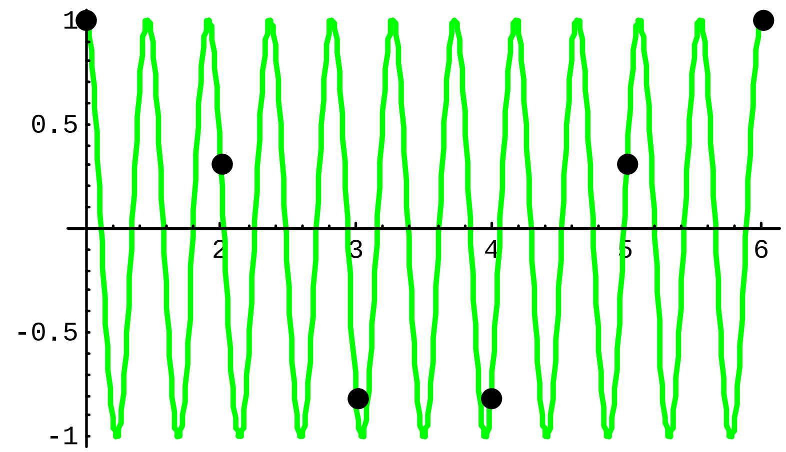

```
In[19]:= p2 = Plot @Cos @2 Pi 0.2 Hx - 1 LD, 8x, 1, 6<,
           8PlotPoints -> 512 ,
            PlotStyle -> 8Thickness @0.01 D, RGBColor @1 , 0, 0 D<<D
```

```
In[11]:= p4 = ListPlot @Table @Cos @2 Pi 1.2 x D, 8x, 0, 5, 1 <D,
           Prolog -> AbsolutePointSize @10 DD
```

```
In[18]:= p1 = Plot @Cos @2 Pi 1.2 Hx - 1 LD, 8x, 1, 6<,
           8PlotPoints -> 512 ,
            PlotStyle -> 8Thickness @0.01 D, RGBColor @0 , 0, 1 D<<D
```

# Sampling



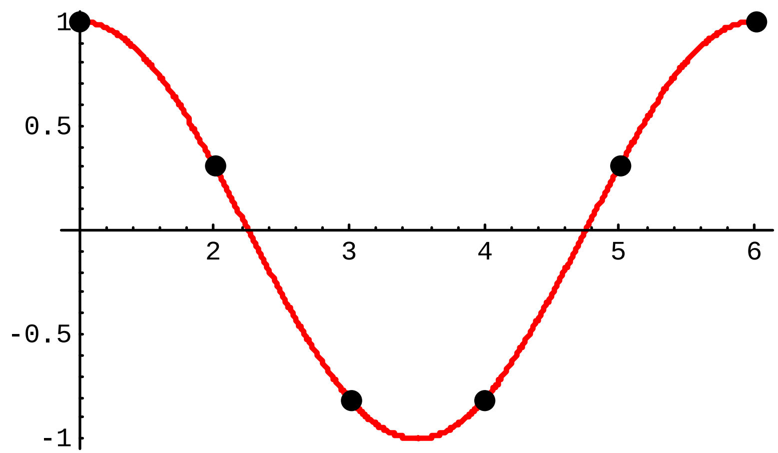



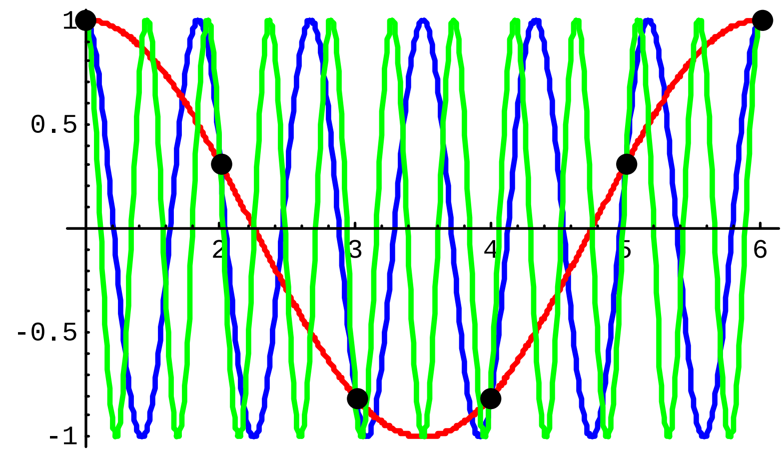

```
In[19]:= p2 = Plot @Cos @2 Pi 0.2 Hx - 1 LD, 8x, 1, 6<,
    8PlotPoints -> 512 ,
     PlotStyle -> 8Thickness @0.01 D, RGBColor @1 , 0, 0 D<<D
```

```
In[11]:= p4 = ListPlot @Table @Cos @2 Pi 1.2 x D, 8x, 0, 5, 1 <D,
    Prolog -> AbsolutePointSize @10 DD
```

```
In[20]:= p3 = Plot @Cos @2 Pi 2.2 Hx - 1 LD, 8x, 1, 6<,
    8PlotPoints -> 512 ,
     PlotStyle -> 8Thickness @0.01 D, RGBColor @0 , 1, 0 D<<D
```

# Sampling




**Nyquist theorem: to correctly identify a frequency you must sample twice a period.**

**So, if $\Delta x$ is the sampling, then $\pi/\Delta x$ is the maximum spatial frequency.**

# Sampling A Simple Cosine Function

Consider a simple cosine function,

$$s_1(t) = \cos(2\pi f_o t)$$

We know the Fourier Transform of this

$$\cos(2\pi f_o t) \Leftrightarrow \pi\left(\delta(f - f_o) + \delta(f + f_o)\right)$$

What happens when we sample this at a rate of $\left(\frac{1}{\Delta t}\right)$

$$where \;\; \frac{1}{\Delta t} \;\; has \;\; the \;\; units \;\; of \;\; Hz$$

$$and \;\; \Delta t = the \;\; \underline{\underline{dwell}} \;\; of \;\; the \;\; sampled \;\; signal.$$

## Sampling A Simple Cosine Function (cont …)

So that,

$$
\begin{aligned}
s_1(n) &= \cos(2\pi f_o n\Delta t) \\
&= s_1(t)\sum \delta(t - n\Delta t) \\
F\{s_1(n)\} &= F\{s_1(t)\sum \delta(t - n\Delta t)\} \\
&= F\{s_1(t)\} \otimes F\{\sum \delta(t - n\Delta t)\} \\
&= \pi\left[\delta(f - f_o) + \delta(f + f_o)\right] \otimes \left[\frac{2\pi}{\Delta t}\sum \delta\left(f - \frac{2\pi n}{\Delta t}\right)\right]
\end{aligned}
$$

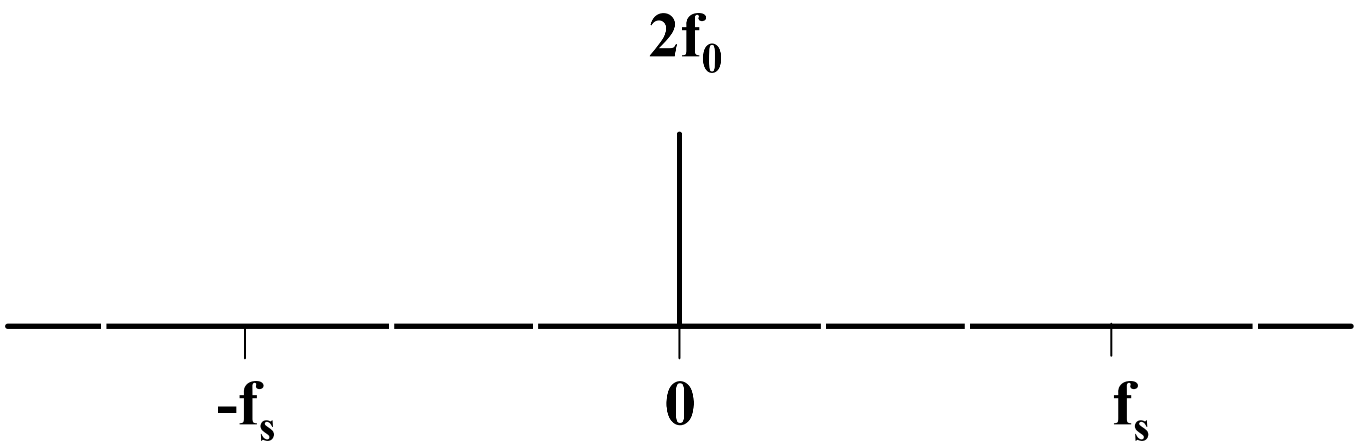
**Note, a short cut has been taken, and left off the integrals that are needed to sample with the delta function.**

# Define A New Frequency: Case 1

Define a new frequency $$f_s = \frac{2\pi}{\Delta t}; \text{"}\textit{the sampling frequency}\text{"}$$

Case 1:

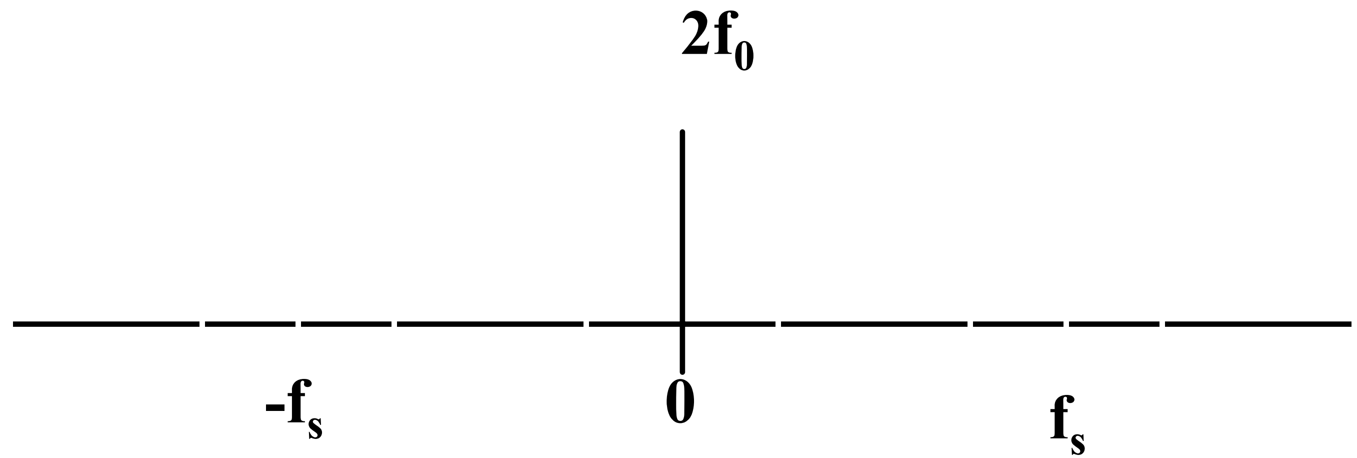
$$f_o < {f_s}/{2} = f_n = Nyquist\ \ frequency$$




In this case, there is no overlap and regardless of the complexity of this spectrum (think of having a number or continuum of cosine functions), the frequency spectrum correctly portrays the time evolution of the signal.

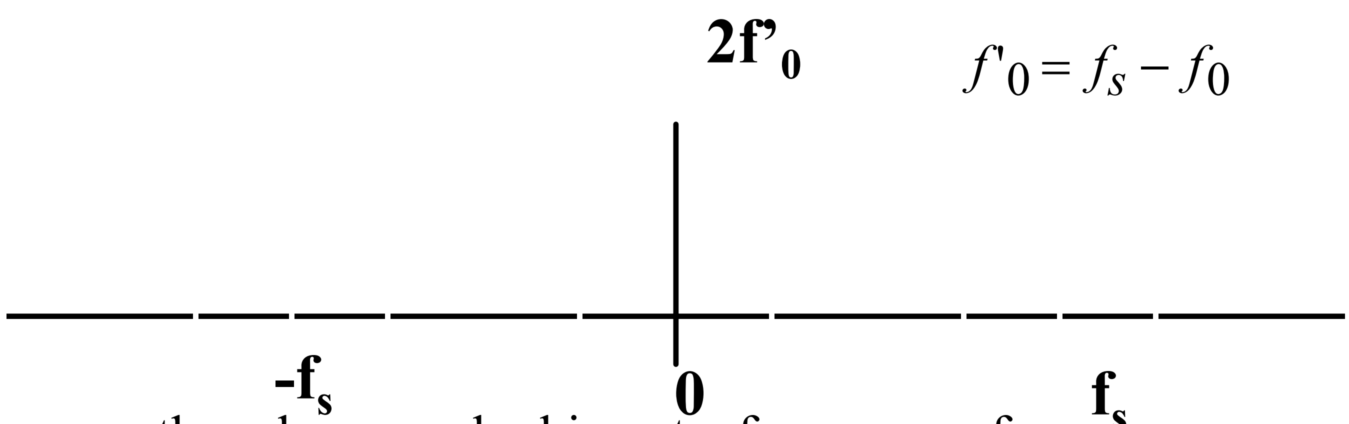
# Case 2

$$f_o > f_s/2 = f_n = Nyquist \quad frequency$$




Now the frequency spectrum overlaps and look what happens to our picture,

$$f'_0 = f_s - f_0$$




So it appears as though we are looking at a frequency of

$f_s - f_o < f_o$ - this is an aliased signal.

# Nyquist Theorem

Consider what happens when there is a complex spectrum. Then the entire spectrum overlaps. Either way, the frequency spectrum does not correspond to a correct picture of the dynamics of the original time domain signal.

| |
|---|
| Nyquist Theorem: In order to correctly determine the frequency spectrum of a signal, the signal must be measured at least twice per period. |

# Return to the sampled cosine function

$$\cos(2\pi f_o n\Delta t)\ ;\ \Delta t = {}^{1}\!/_{f_s}$$

So

$$\cos\left(\frac{2\pi f_o n}{f_s}\right)$$

Now let

$$f_o > f_s \quad \therefore \quad f_o = f_s - \underbrace{(f_s - f_o)}_{\Delta f}$$

## (cont…)

$$\cos\left(\frac{2\pi n}{f_s}[f_s - \Delta f]\right)$$

$$\cos\left(2\pi n - \frac{2\pi n}{f_s}\Delta f\right)$$

$$\cos\left(-2\pi n\frac{\Delta f}{f_s}\right)$$

*or use:*

$\cos(A+B) =$

$\cos(A)\cos(B) - \sin(A)\sin(B)$

$A = 2\pi n \quad \therefore \quad \cos(A) = 1; \sin(A) = 0$

$$\boxed{\cos\left(\frac{2\pi n\Delta f}{f_s}\right)}$$

Therefore we can see that it is not the Fourier Transform that fails to correctly portray the signal, but by our own sampling process we mis-represented the signal.

# A Fourier Picture of Sampling

$$\{f_n\} = \int_{-\infty}^{\infty} \underbrace{TopHat\left(\frac{2x}{fov}\right)}_{\Updownarrow} \underbrace{\sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)}_{\Updownarrow} \underbrace{f(x)}_{\Updownarrow} dx$$

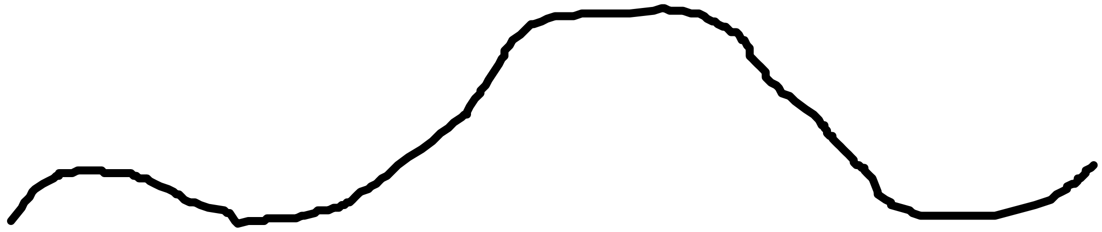
$$Sinc(kfov) \otimes \underbrace{2\pi\Delta x \sum_{n=-\infty}^{\infty} \delta\left(k - \frac{n2\pi}{\Delta x}\right)}_{Look\ \ at\ \ this\ \ first} \otimes F(k)$$

# A Fourier Picture of Sampling

# Bandwidth Limited

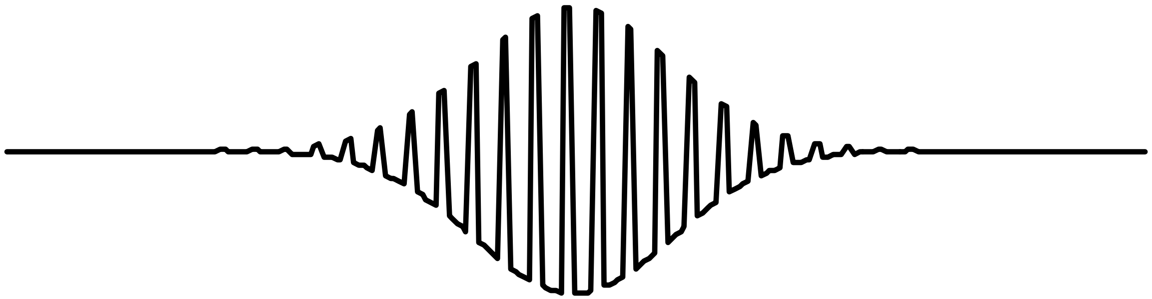
## Time Domain

# Limitations of Sampling

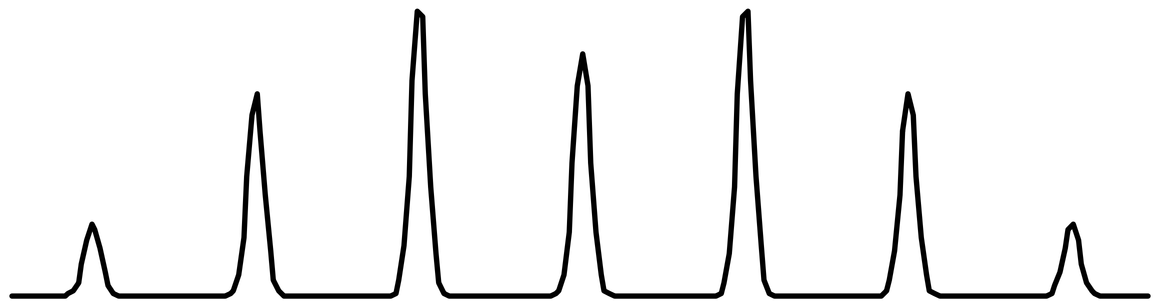


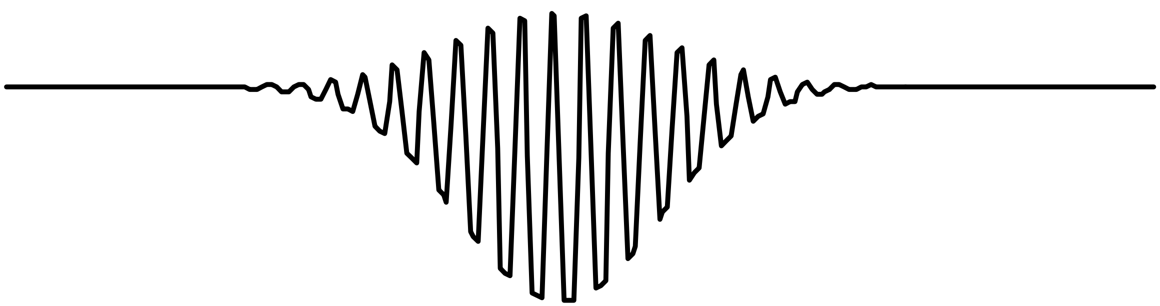
$\{f_n\}$

# Limitations of Sampling

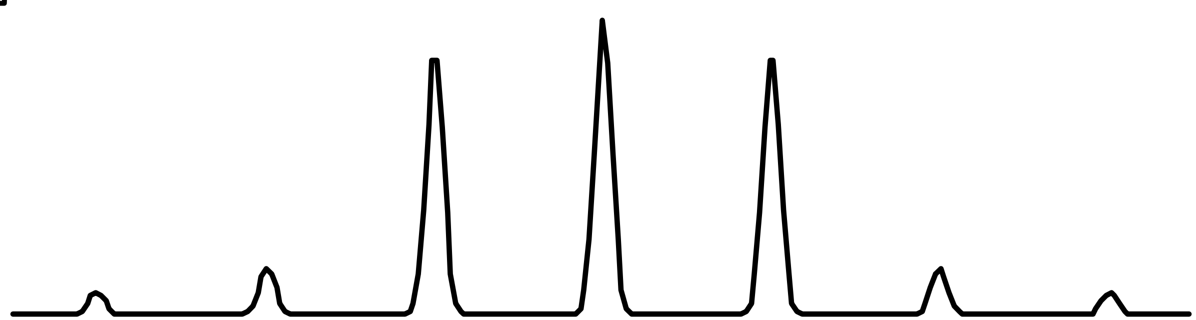


**perfect sampling**

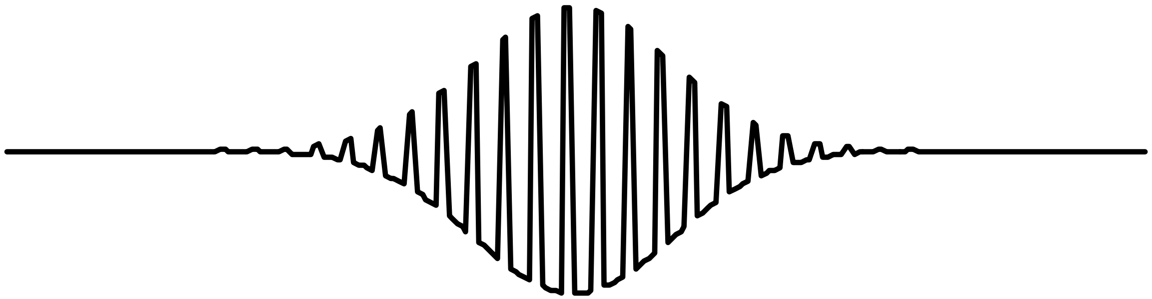




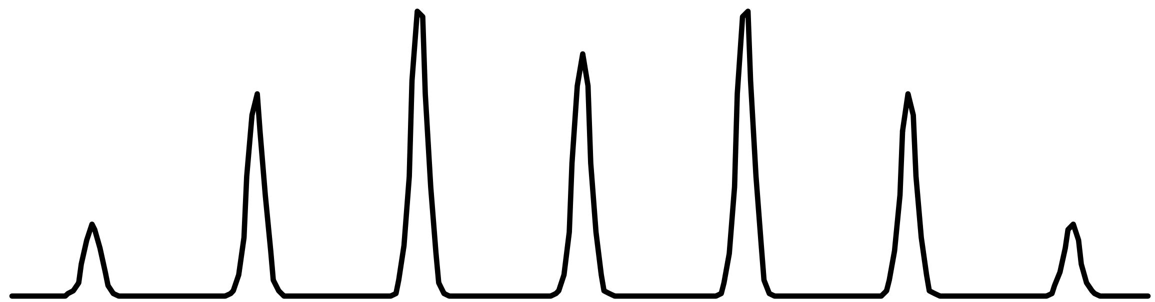
**average 3 data points**

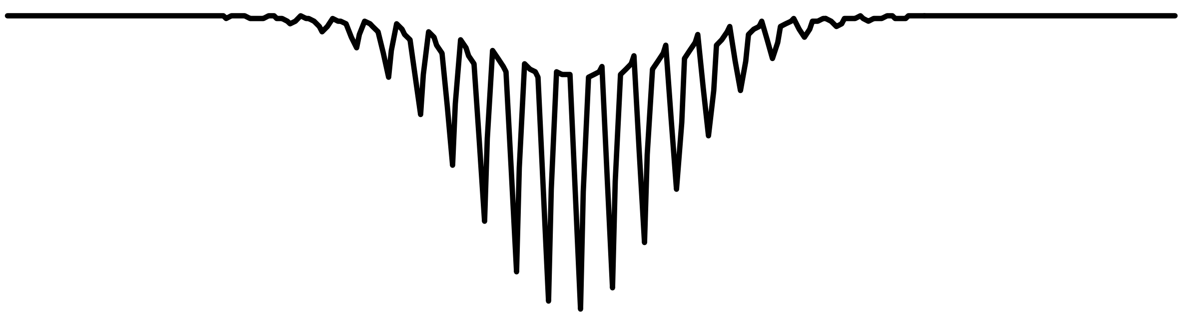


# Limitations of Sampling

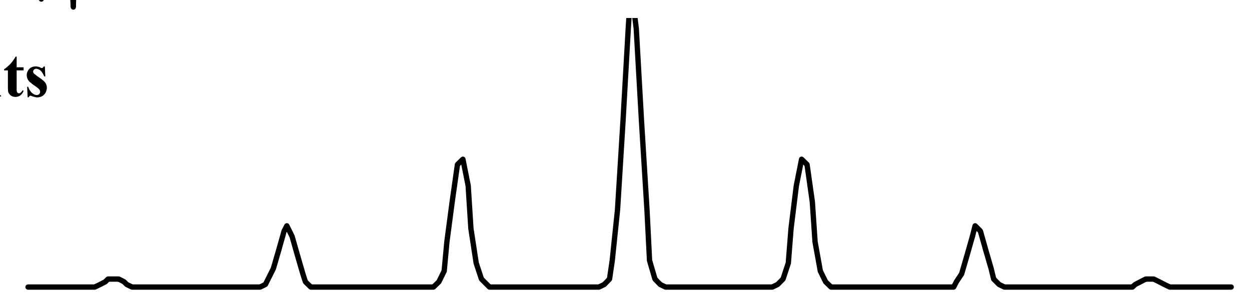


**perfect sampling**

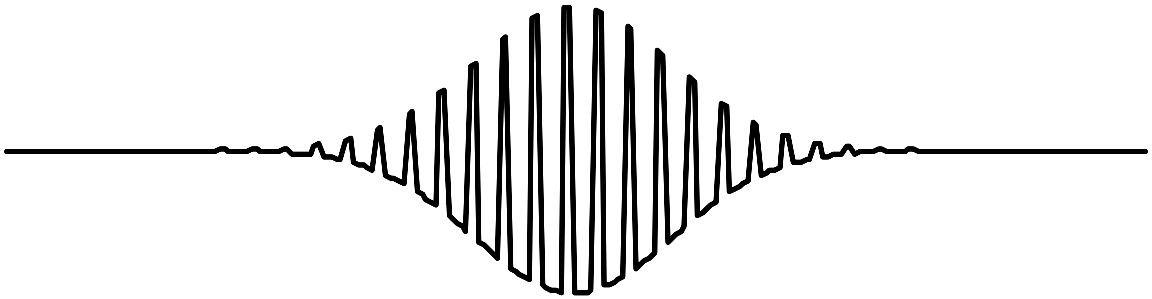




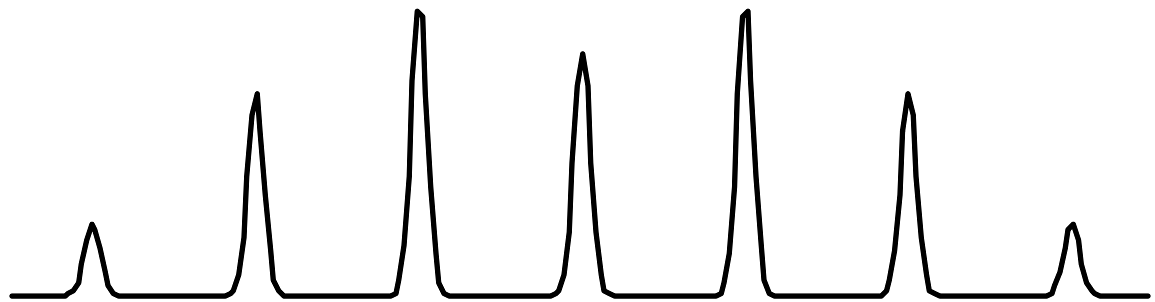
**average 5 data points**

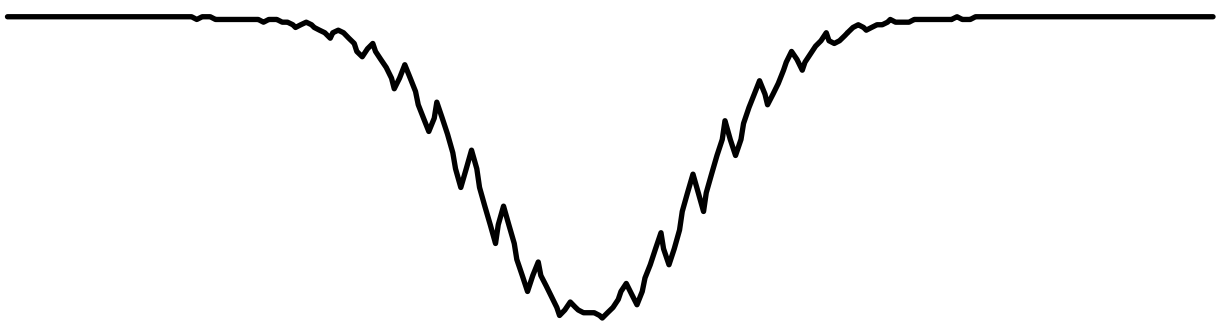


# Limitations of Sampling

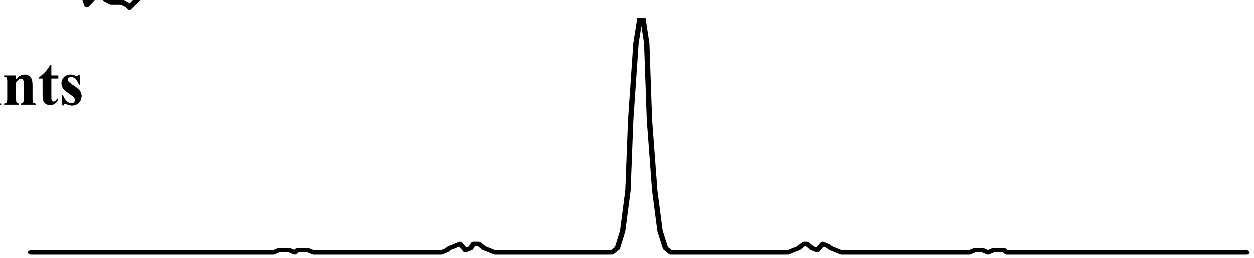


**perfect sampling**

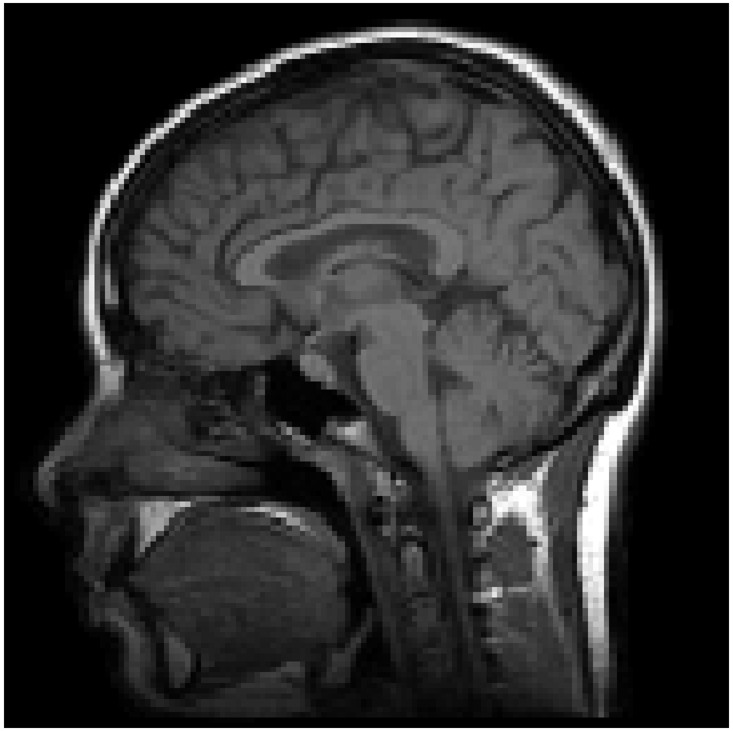




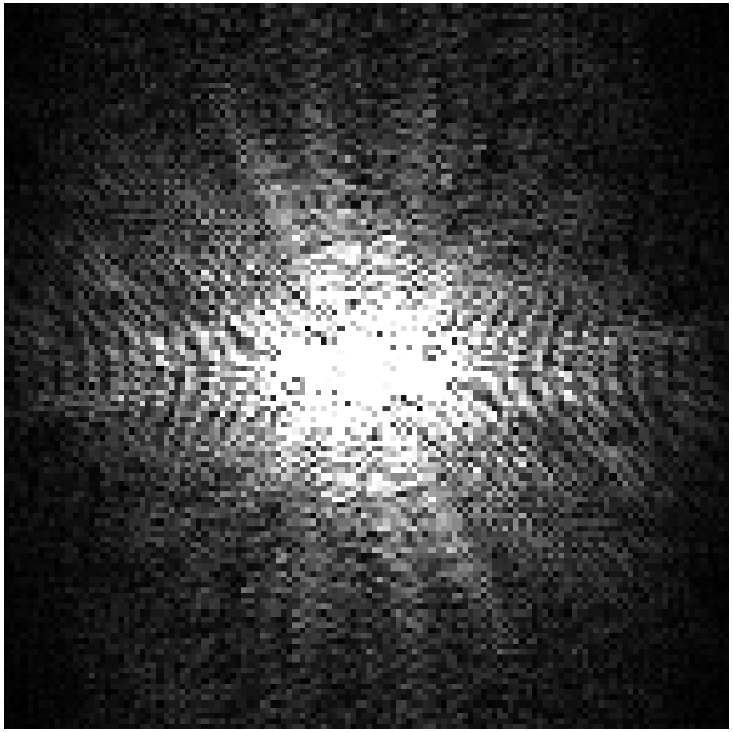
**average 7 data points**



# Reciprocal Space

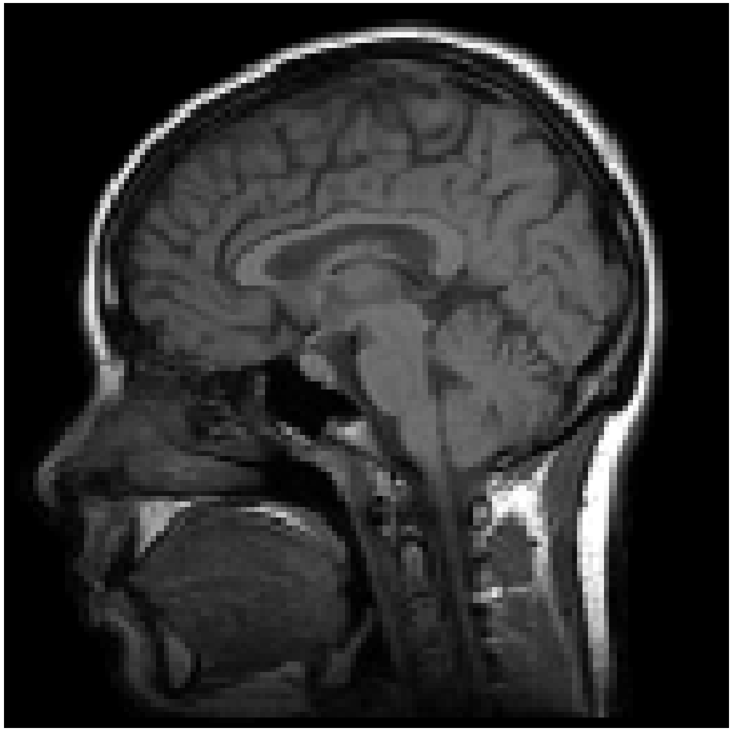


real space

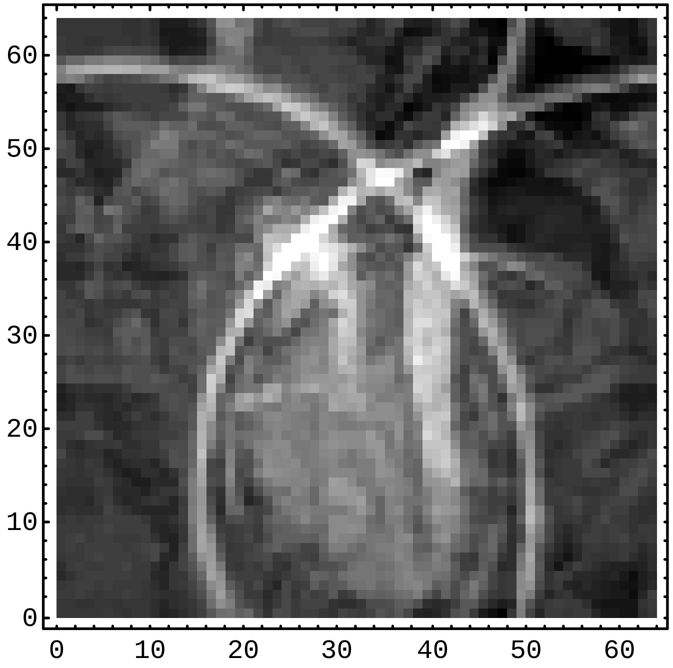


reciprocal space

# Sampling

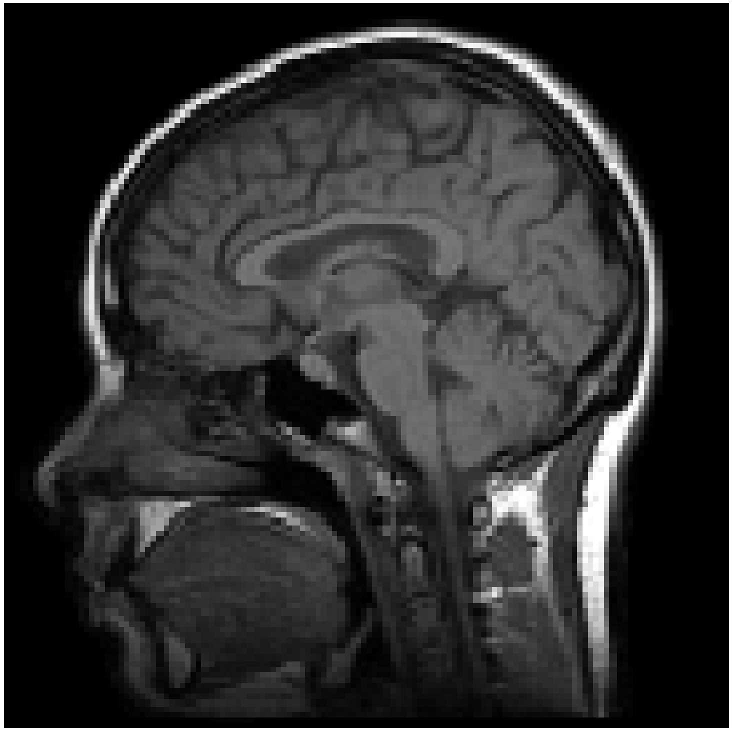


real space

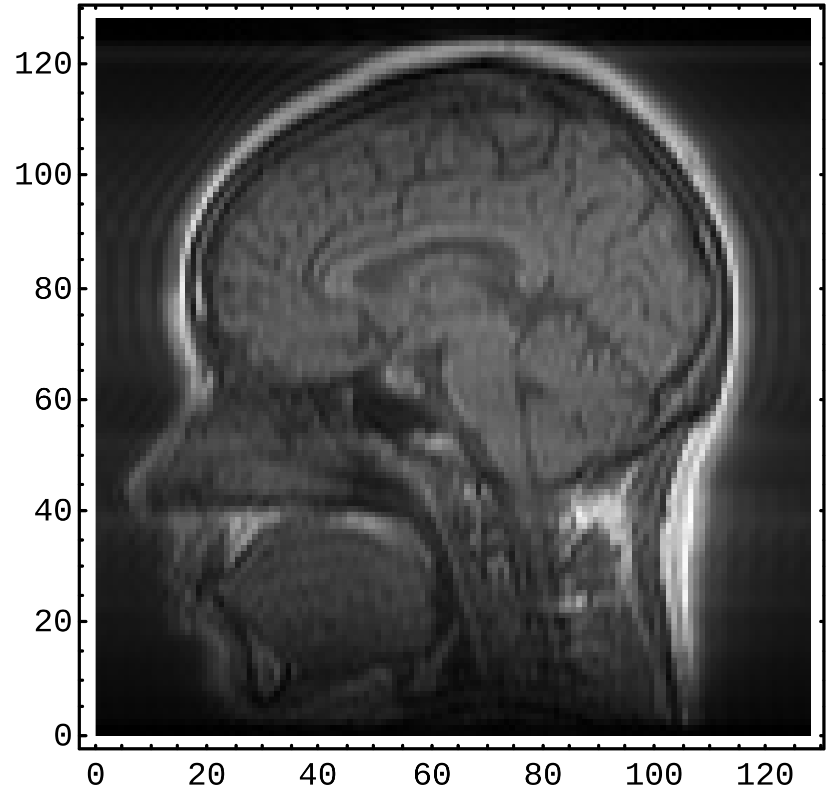



under sampled

# Sampling

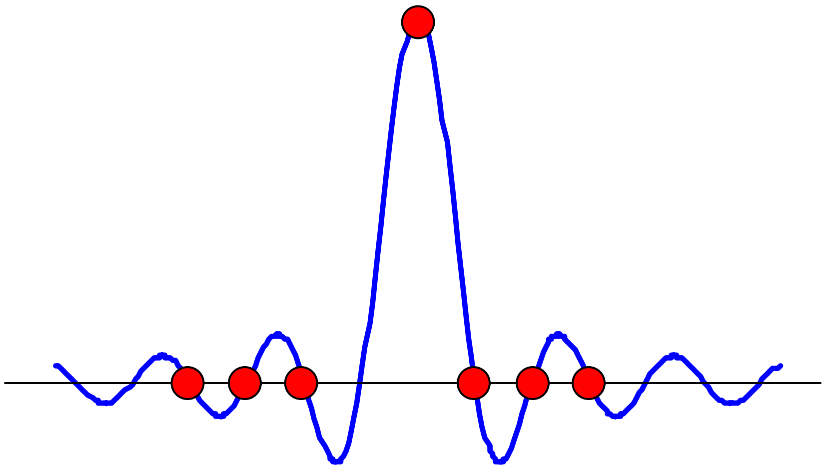


real space



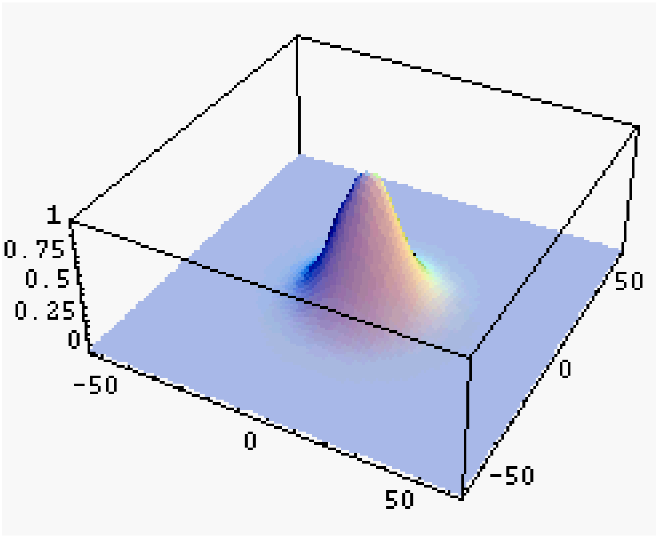

zero filled

# Sinc interpolation

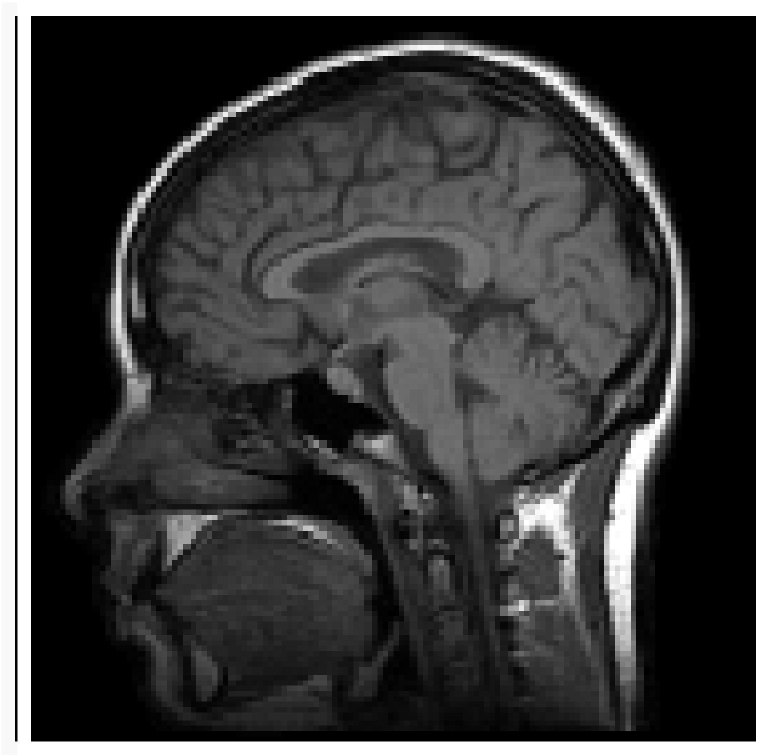


# Filtering

We can change the information content in the image by manipulating the information in reciprocal space.

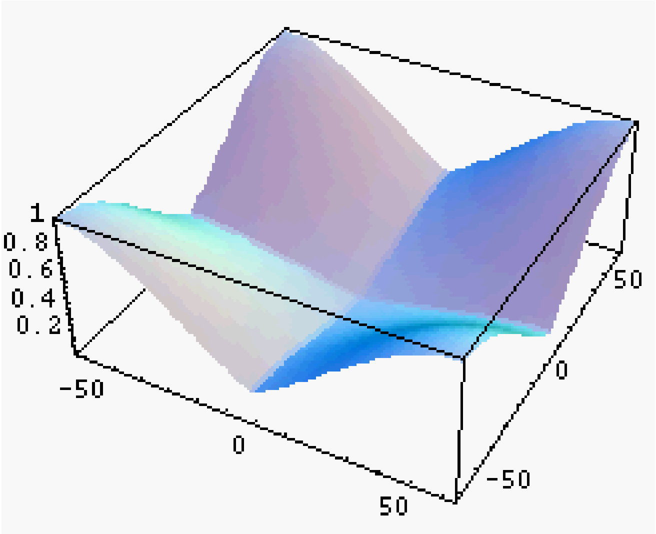



Weighting function in k-space.

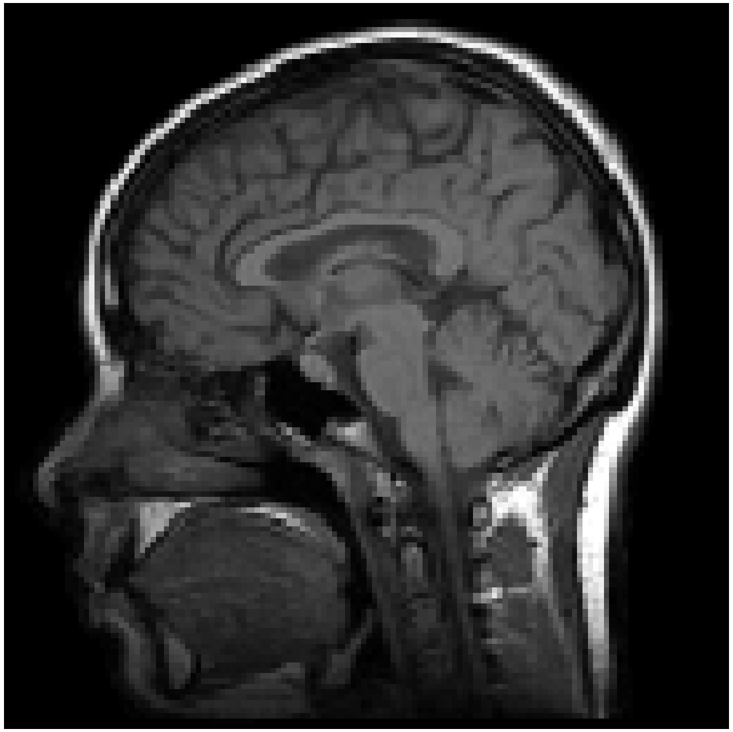


# Filtering

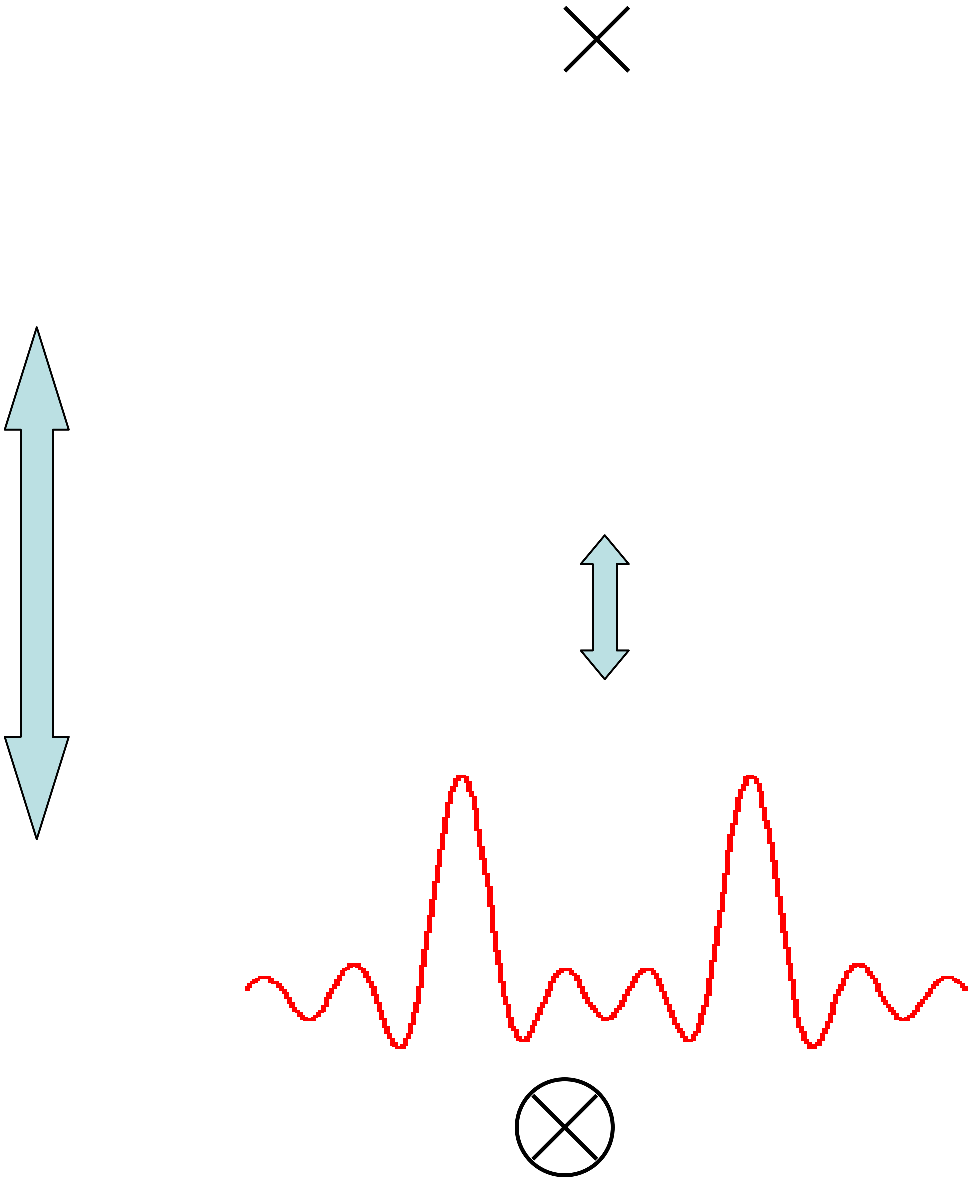
We can also emphasis the high frequency components.




Weighting function in k-space.



# Fourier Convolution



# Deconvolution

$$I(x) = O(x) \otimes PSF(x) + \underbrace{N}_{noise}(x)$$

Recall in an ideal world

$$i(k) = O(k) - PSF(k)$$

$\therefore$ Try deconvolution with $1/PSF(k)$

$$i_{perfect}(k) = [O(k) \bullet PSF(k)] \bullet \frac{1}{PSF(k)}$$

*an "inverse" filter*

$$I_{perfect}(x) = [O(x) \otimes PSF(x) + N(x)] \otimes \underbrace{PSF^{-1}(x)}_{\substack{means\ "inverse" \\ not\ 1/PSF(x)}}$$

# Deconvolution (cont…)

This appears to be a well-balanced function, but look what happens in a Fourier space however:

$$i(k) = O(k) \bullet PSF(k) \bullet \frac{1}{PSF(k)} + \frac{n(k)}{PSF(k)}$$

Recall that the Fourier Transform is linear where PSF(k) << n(k), then noise is blown up. The inverse filter is ill-conditioned and greatly increases the noise particularly the high-frequency noise since

$$\underbrace{PSF(k) \Rightarrow 0 \quad at \quad high \quad k}_{normally}$$

This is avoided by employing a "Wiener" filter.

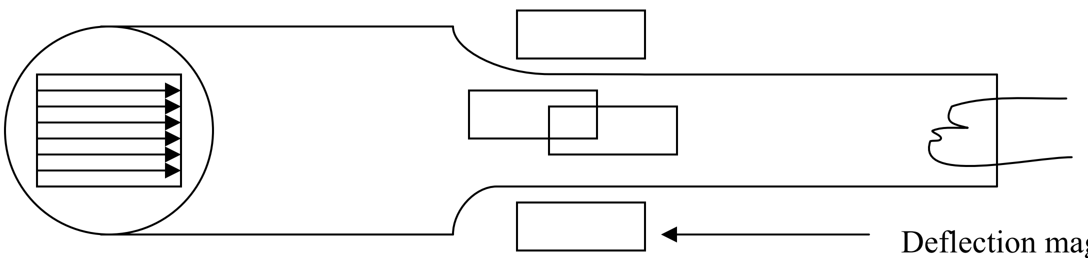
$$PSF_w(k) = \frac{PSF^*(k)}{\left|PSF(k)\right|^2 + W_N(k)} \quad where * is \quad the \quad complex \quad conjugate$$

*noise power spectral density* $= \left(S/N(k)\right)^{-2}$

$\left(S/N\right)^{-2}$

# Original Nyquist Problem

A black & white TV has 500 lines with 650 elements per line. These are scanned through




Electron beam is scanned over the screen by deflection magnets and the intensity of the beam is modulated to give the intensity at each point. The refresh rate is 30 frames/second.

By the sampling theorem, if the frequency is $f_o$, independent information is available once every $\frac{1}{2f}$ seconds.

Need information $$30\,\frac{frames}{\text{sec}} \times 500\,\frac{lines}{frame} \times 650\,\frac{pixels}{line} = 9.75 \times 10^6\ impulse/\text{sec}$$

$$\therefore f \geq 4.875 \quad MHz$$

# Fourier Transform of a Comb Function

$$Comb(x) = \sum_{-\infty}^{\infty} \delta(x - nX)$$

*where*

$$\delta(x - nX) = \begin{Bmatrix} 1, & x = nX \\ 0, & x \neq nX \end{Bmatrix}$$

The Fourier Transform we wish to evaluate is,

$$F\{Comb(t)\} = \int_{-\infty}^{\infty} Comb(x) e^{-ikx} dx$$

One trick to this is to express the comb function as a Fourier series expansion, not the transform.

$$Comb(x) = \sum_{n=-\infty}^{\infty} a_n e^{i2\pi nx/X}$$

*where*

$$a_n = \frac{1}{X} \int_{-x/2}^{x/2} Comb(x) e^{-i2\pi nx/X} dx$$

Normalization to keep the series unitary

Limits chosen since this is a periodic function

# Fourier Transform of a Comb Function

Over the interval $-x/2$ to $x/2$ the comb function contains only a single delta function.

$$a_n = \frac{1}{X}\underbrace{\int_{-x/2}^{x/2} \delta(x) e^{-i2\pi nx/X}\,dx}_{\substack{select\ the\ x=0\ point,\\ note\ this\ is\ only\ true\\ of\ a\ Dirac\ delta\ function.}}$$

$$a_n = \frac{1}{X}$$

$\therefore$ A series representation of the comb function is

$$\boxed{Comb(x) = \frac{1}{X}\sum_{n=-\infty}^{\infty} e^{i2\pi nx/X}}$$

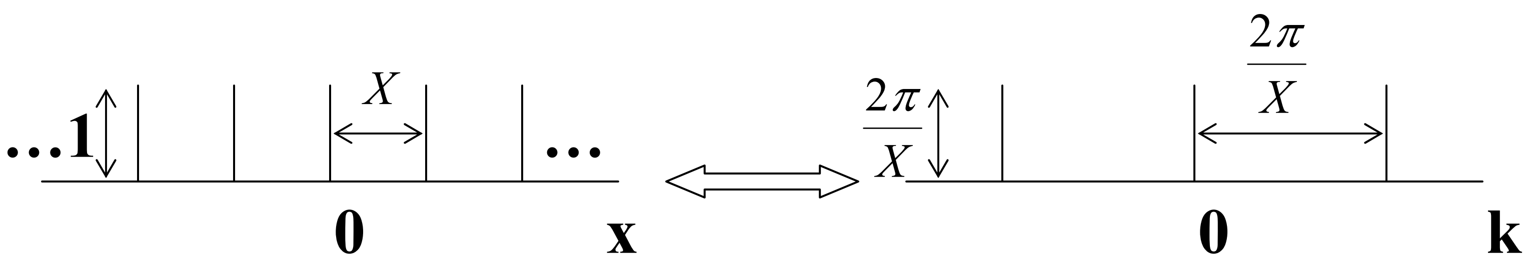
# Fourier Transform of a Comb Function

Now,

$$F\{Comb\} = \int_{-\infty}^{\infty} \frac{1}{X} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/X} e^{-ikx} dx$$

$$= \frac{1}{X} \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{-ix\left(k-\frac{2\pi n}{X}\right)} dx}_{2\pi\delta\left(k-\frac{2\pi n}{X}\right)}$$

$$\sum_{-\infty}^{\infty} \delta(x-nX) \Leftrightarrow \frac{2\pi}{X} \sum_{-\infty}^{\infty} \delta\left(k-\frac{2\pi n}{X}\right)$$




Note: Scaling laws hold