Exercise 5. Diffusion and Parabolic Equations.

1. Write a computer code to solve the diffusive equation

\[ \frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2} + s(x) \]

For constant, uniform diffusivity \( D \) and constant specified source \( s(x) \). Use a uniform \( x \)-mesh with \( N_x \) nodes. Consider boundary conditions to be \( \psi = \psi_1 \) at \( x = 0 \) and \( \frac{\partial \psi}{\partial x} = 0 \) at \( x = 1 \) (the domain boundaries).

Construct a matrix \( D = D_{ij} \) such that \( D \psi = \nabla^2 \psi \). Use it to implement the FTCS scheme

\[ \psi^{(n+1)} = (I + \Delta tD)\psi^{(n)} + \Delta t s, \]

paying special attention to the boundary conditions.

Solve the time-dependent problem, for \( t = 0 \rightarrow 1 \), when \( D = 1, s = 1, N_x = 50, \psi_1 = 0 \), with initial condition \( \psi = 0 \) at \( t = 0 \) storing your results in a matrix \( \psi(x,t) = \psi_{jx,jt} \), and plotting that matrix at the end of the solution, for examination.

Experiment with various \( \Delta t \) to establish the dependence of the accuracy and stability of your solution on \( \Delta t \). In particular,

(i) find experimentally the value of \( \Delta t \) above which the scheme becomes unstable.
(ii) estimate experimentally the value of \( \Delta t \) at which \( \psi(t=1) \) is accurate to 1%.

2. Develop a modified version of your code to implement the \( \theta \)-implicit scheme:

\[ (I - \Delta t\theta D)\psi^{(n+1)} = (I + \Delta t(1 - \theta)D)\psi^{(n)} + \Delta t s, \]

in the form

\[ \psi^{(n+1)} = B^{-1}C\psi^{(n)} + B^{-1}\Delta t s \]

Experiment with \( \Delta t \) and different \( \theta \) values, for the same time-dependent problem and find experimentally the value of \( \theta \) for which instability disappears for all \( \Delta t \).

Also choose a \( \Delta t \) value for which the FTCS (\( \theta = 0 \)) scheme is stable; then find experimentally the approximate optimum value of \( \theta \) (at that fixed \( \Delta t \)) which produces the most accurate results.

Submit the following as your solution for each part:

a. Your code in a computer format that is capable of being executed, citing the language it is written in.

b. The requested experimental \( \Delta t \) and/or \( \theta \) values.

c. A plot of your solution for at least one of the cases.

d. A brief description of how you determined the accuracy of the result.