Reading: Chapters 2 & 3 of Sigmar & Helander

1. Equilibration: Section 3.3 in the book considers collisions of test particles with a Maxwellian field particle distribution. The result in eq. (3.40) of the book involves collision frequencies, $\nu_{sb}^{ab}(v)$ and, $\nu_{\parallel}^{ab}(v)$ and it is not obvious that a Maxwellian will result for the test particles in equilibrium. Consider identical field and test particles, so that, $m_a = m_b$. Show that actually, 

$$\frac{\nu_{sb}^{ab}(v)}{\nu_{\parallel}^{ab}(v)} = \frac{2v}{v_T^2}$$

You may find equations, 3.45-3.48 helpful for this. Now you can write the velocity magnitude part of the operator as,

$$C_v \equiv \frac{1}{2v^2} \frac{\partial}{\partial v} v^4 \nu_{\parallel}(v) \left( \frac{2v}{v_T} f + \frac{\partial f}{\partial v} \right)$$

This is now analogous to the 1D example we looked at in lecture, except for the magnitude of velocity, $v$, in a 3D velocity space. Show that for, $C_v \rightarrow 0$, the distribution goes to a Maxwellian, $f \rightarrow f_M$.

2. Fokker-Planck equation accuracy: Considering the Fokker-Planck equation as a Taylor series expansion, we could continue to higher order as follows,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v} \cdot A f + \frac{\partial^2}{\partial v \partial v} : D f + \frac{\partial^3}{\partial v \partial v \partial v} : T f$$

where, $T$, is some rank 3 tensor. Make a simple scaling argument on the coefficients (assuming the small angle expansion) to show that the terms in $T$ (and higher order terms) are order unity compared to the divergent, $\sim \ln \Lambda$, terms retained in the Fokker-Planck equation. Estimate from this the inherent error in the Fokker-Planck operator. You may find some helpful arguments in the book for this problem.

3. Collision Operator Properties: Prove conservation of mass, momentum, and energy first for the single species collision operator, and then for a 2 species system consisting of electrons (subscript, $e$), and a single species of ions (subscript, $i$).

4. H-Theorem: Prove the H-theorem as follows:

Show that the rate of change of entropy is given by,

$$\frac{dS}{dt} = -\frac{d}{dt} \int d^3v f \ln f = -\int d^3v \ln f C(f, f)$$
By appropriate manipulations (integration by parts, reversing dummy variables, etc.) work this into the expression,

\[
\frac{dS}{dt} = \frac{1}{2} \Gamma \int d^3v d^3v' f(v) f(v') \left( \frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right) \cdot \mathbf{U} \cdot \left( \frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right)
\]

where, \( f' = f(v') \).

Show that, \( c \cdot \mathbf{U} \cdot c = |\mathbf{u} \times c|^2 / u^3 > 0 \), for any vector, \( c \). It now follows that,

\[
\frac{dS}{dt} \geq 0
\]

Why?

\( dS/dt = 0 \) if and only if, \( \mathbf{u} \times c = 0 \), and this must hold for all, \( v \) and \( v' \). Show then that this implies,

\[
(v - v') \times \left( \frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right) = 0
\]

and that this implies that \( f \) must be Maxwellian, \( f = \text{const.} \exp \left( - (v - V)^2 / v_T^2 \right) \). Here, \( V \), is some constant, fluid, velocity.

5. **Positivity**: Show that, \( f > 0 \), at \( t = 0 \), implies, \( f > 0 \), for all times.