8.022 (E&M) – Lecture 4

Topics:
- More applications of vector calculus to electrostatics:
  - Laplacian: Poisson and Laplace equation
  - Curl: concept and applications to electrostatics
- Introduction to conductors

Last time...
- Electric potential: \[ \phi(\vec{r}) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s} \] with \( \vec{E} = -\nabla \phi \)
  - Work done to move a unit charge from infinity to the point \( P(x,y,z) \)
  - It's a scalar!
- Energy associated with an electric field:
  - Work done to assemble system of charges is stored in \( E \)
    \[ U = \frac{1}{2} \int_{\text{Volume with charges}} \rho \phi(r) dV = \int_{\text{Entire space}} \frac{E^2}{8\pi} dV \]
- Gauss's law in differential form: \( \nabla \cdot \vec{E} = 4\pi \rho \)
  - Easy way to go from \( E \) to charge distribution that created it
Laplacian operator

What if we combine gradient and divergence? Let’s calculate the \( \nabla \cdot \nabla f \) (Q: difference wrt \( \text{grad div} \ f \) ?)

\[
\nabla \cdot \nabla f = \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \left( \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \right)
\]

\[
= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f = \nabla^2 f
\]

\[
\nabla^2 f \equiv \nabla \cdot \nabla f \quad \text{Laplace Operator}
\]

Interpretation of Laplacian

Given a 2d function \( \phi(x,y) = a(x^2 + y^2) / 4 \) calculate the Laplacian

\[
\nabla^2 f = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f = a \frac{a}{4} (2 + 2) = a
\]

As the second derivative, the Laplacian gives the curvature of the function
Poisson equation

Let’s apply the concept of Laplacian to electrostatics.

- Rewrite Gauss’s law in terms of the potential

\[
\begin{align*}
\nabla \cdot \vec{E} &= 4\pi \rho \\
\nabla \cdot \vec{E} &= \nabla \cdot (-\nabla \phi) = -\nabla^2 \phi
\end{align*}
\]

\[\nabla^2 \phi = -4\pi \rho \quad \text{Poisson Equation}\]

Laplace equation and Earnshaw’s Theorem

- What happens to Poisson’s equation in vacuum?

\[\nabla^2 \phi = -4\pi \rho \Rightarrow \nabla^2 \phi = 0 \quad \text{Laplace Equation}\]

- What does this teach us?
  In a region where \( \phi \) satisfies Laplace’s equation, then its curvature must be 0 everywhere in the region.
  \[\Rightarrow \text{The potential has no local maxima or minima in that region}\]

- Important consequence for physics:
  Earnshaw’s Theorem:
  It is impossible to hold a charge in stable equilibrium with electrostatic fields (no minima)
Application of Earnshaw’s Theorem

8 charges on a cube and one free in the middle. Is the equilibrium stable? No!

(does the question sound familiar?)

The circulation

- Consider the line integral of a vector function $\mathbf{F}$ over a closed path $C$:

$$\Gamma = \oint_{C} \mathbf{F} \cdot d\mathbf{s}$$

- Let’s now cut $C$ into 2 smaller loops: $C_1$ and $C_2$
- Let’s write the circulation $C$ in terms of the integral on $C_1$ and $C_2$

$$\Gamma = \oint_{C} \mathbf{F} \cdot d\mathbf{s} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} =$$

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{s} - \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} + \oint_{C_1} \mathbf{F} \cdot d\mathbf{s}$$

$$= \oint_{C_1} \mathbf{F} \cdot d\mathbf{s} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} \implies \Gamma = \Gamma_1 + \Gamma_2$$
The curl of \( \mathbf{F} \)

- If we repeat the procedure \( N \) times:
  \[
  \Gamma = \sum_{i=1}^{i=\text{Large}N} \Gamma_i
  \]

- Define the curl of \( \mathbf{F} \) as circulation of \( \mathbf{F} \) per unit area in the limit \( A \to 0 \)
  \[
  \text{curl } \mathbf{F} \cdot \mathbf{n} \equiv \lim_{A \to 0} \frac{\oint_C \mathbf{F} \cdot d\mathbf{s}}{A}
  \]
  where \( A \) is the area inside \( C \)

- The curl is a vector normal to the surface \( A \) with direction given by the "right hand rule"

---

Stokes Theorem

\[
\Gamma = \sum_{i=1}^{i=\text{Large}N} \Gamma_i = \sum_{i=1}^{\text{Large}N} \oint_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{\text{Large}N} A_i \oint_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{\text{Large}N} A_i \frac{\oint_C \mathbf{F} \cdot d\mathbf{s}}{A_i}
\]

In the limit \( A \to 0 \):

\[
\oint_C \mathbf{F} \cdot d\mathbf{s} \to \text{curl } \mathbf{F} \cdot \mathbf{n} \quad \text{and} \quad \sum_{i=1}^{\text{Large}N} A_i \to \int_A dA
\]

\[
\Gamma = \sum_{i=1}^{\text{Large}N} A_i \text{ curl } \mathbf{F} \cdot \mathbf{n} = \sum_{i=1}^{\text{Large}N} \text{ curl } \mathbf{F} \cdot (A_i \mathbf{n}) = \sum_{i=1}^{\text{Large}N} \text{ curl } \mathbf{F} \cdot A_i \to \int_A \text{ curl } \mathbf{F} \cdot dA
\]

\[
\Gamma = \oint_C \mathbf{F} \cdot d\mathbf{s} \quad \text{(definition of circulation)}
\]

\[
\Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_A \text{ curl } \mathbf{F} \cdot dA
\]

**Stokes Theorem**

NB: Stokes relates the line integral of a function \( \mathbf{F} \) over a closed line \( C \) and the surface integral of the curl of the function over the area enclosed by \( C \)
Application of Stoke’s Theorem

- Stoke’s theorem:
  \[
  \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_A \text{curl} \mathbf{F} \cdot d\mathbf{A}
  \]

- The Electrostatics Force is conservative:
  \[
  \oint_C \mathbf{F} \cdot d\mathbf{s} = 0
  \]
  \[
  \Rightarrow \int_A \text{curl} \mathbf{E} \cdot d\mathbf{A} = 0 \quad \text{for any surface} \ A
  \]
  \[
  \Rightarrow \text{curl} \mathbf{E} = 0
  \]

- The curl of an electrostatic field is zero.

Curl in cartesian coordinates (1)

- Consider infinitesimal rectangle in yz plane centered at \(P=(x,y,z)\) in a vector field \(\mathbf{F}\)
- Calculate circulation of \(\mathbf{F}\) around the square:
  \[
  \int \mathbf{F} \cdot d\mathbf{s} = F_x(x, y + \frac{\Delta y}{2}, z) \Delta y - F_x(x, y - \frac{\Delta y}{2}, z) \Delta y
  \]
  \[
  \int \mathbf{F} \cdot d\mathbf{s} = F_y(x, y + \frac{\Delta y}{2}, z) \Delta z - F_y(x, y - \frac{\Delta y}{2}, z) \Delta z
  \]
  \[
  \int \mathbf{F} \cdot d\mathbf{s} = F_z(x, y, z + \frac{\Delta y}{2}) \Delta y - F_z(x, y, z - \frac{\Delta y}{2}) \Delta y
  \]
  \[
  \int \mathbf{F} \cdot d\mathbf{s} = F_z(x, y, z + \frac{\Delta z}{2}) \Delta y - F_z(x, y, z - \frac{\Delta z}{2}) \Delta y
  \]

Adding the 4 components:
  \[
  \oint \mathbf{F} \cdot d\mathbf{s} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \Delta y \Delta z
  \]
Curl in cartesian coordinates (2)

Combining this result with definition of curl:

\[
\text{curl } \vec{F} = \lim_{\Delta A \to 0} \frac{\oint_{\text{square}} \vec{F} \cdot d\vec{s}}{A} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} & \frac{\partial F_x}{\partial x} \\
\frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} & \frac{\partial F_y}{\partial x} \\
\frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} & \frac{\partial F_z}{\partial x}
\end{vmatrix}
\]

\[
\text{(curl } \vec{F}) = \lim_{\Delta y \to 0} \frac{\oint_{\text{square}} \vec{F} \cdot d\vec{s}}{\Delta x \Delta y} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right)
\]

Similar results orienting the rectangles in // (xz) and (xy) planes →

\[
\text{curl } \vec{F} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
\frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} & \frac{\partial F_x}{\partial x} \\
\frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} & \frac{\partial F_y}{\partial x} \\
\frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} & \frac{\partial F_z}{\partial x}
\end{vmatrix}
\]

This is the usable expression for the curl: easy to calculate!

Summary of vector calculus in electrostatics (1)

- **Gradient:** \( \nabla \phi = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \phi \)

  - In E&M: \( \vec{E} = -\nabla \phi \)

- **Divergence:** \( \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \)

  - Gauss's theorem: \( \int \vec{E} \cdot d\vec{A} = \int \nabla \cdot \vec{E} \, dV \)

  - In E&M: Gauss' law in differential form \( \nabla \cdot \vec{E} = 4\pi \rho \)

- **Curl:** \( \text{curl } \vec{F} = \nabla \times \vec{F} \)

  - Stoke's theorem: \( \oint_{\text{c}} \vec{F} \cdot d\vec{s} = \int_{\Delta} \text{curl } \vec{F} \cdot d\vec{A} \)

  - In E&M: \( \nabla \times \vec{E} = 0 \)
Summary of vector calculus in electrostatics (2)

- **Laplacian:** \( \nabla^2 \phi \equiv \nabla \cdot \nabla \phi \)
  - In E&M:
    - Poisson Equation: \( \nabla^2 \phi = -4\pi \rho \)
    - Laplace Equation: \( \nabla^2 \phi = 0 \)
  - Earnshaw’s theorem: impossible to hold a charge in stable equilibrium with electrostatic fields (no local minima)

Comment:
This may look like a lot of math: it is! Time and exercise will help you to learn how to use it in E&M

Conductors and Insulators

**Conductor:** a material with free electrons
- Excellent conductors: metals such as Au, Ag, Cu, Al,...
- OK conductors: ionic solutions such as NaCl in H₂O

**Insulator:** a material without free electrons
- Organic materials: rubber, plastic,...
- Inorganic materials: quartz, glass,...
Electric Fields in Conductors (1)

- A conductor is assumed to have an infinite supply of electric charges
  - Pretty good assumption...
- Inside a conductor, \( E = 0 \)
  - Why? If \( E \) is not 0 \( \rightarrow \) charges will move from where the potential is higher to where the potential is lower; migration will stop only when \( E = 0 \).
  - How long does it take? \( 10^{-17} - 10^{-16} \) s (typical resistivity of metals)

Electric Fields in Conductors (2)

- Electric potential inside a conductor is constant
  - Given 2 points inside the conductor \( P_1 \) and \( P_2 \), the \( \Delta \phi \) would be:
    \[
    \Delta \phi = \int_{P_1}^{P_2} E \cdot dS = 0 \quad \text{since} \quad E = 0 \text{ inside the conductor.}
    \]
- Net charge can only reside on the surface
  - If net charge inside the conductor \( \rightarrow \) Electric Field \( = 0 \) (Gauss’s law)
- External field lines are perpendicular to surface
  - \( E \parallel \) component would cause charge flow on the surface until \( \Delta \phi = 0 \)
- Conductor’s surface is an equipotential
  - Because it’s perpendicular to field lines
Corollary 1

In a hollow region inside conductor, $\phi=\text{const}$ and $E=0$ if there aren’t any charges in the cavity.

Why?

- Surface of conductor is equipotential
- If no charge inside the cavity $\Rightarrow$ Laplace holds $\Rightarrow$ $\phi_{\text{cavity}}$ cannot have max or minima
  $\Rightarrow$ $\phi$ must be constant $\Rightarrow E=0$

Consequence:
- Shielding of external electric fields: Faraday’s cage

Corollary 2

A charge $+Q$ in the cavity will induce a charge $+Q$ on the outside of the conductor.

Why?

- Apply Gauss’s law to surface - - - inside the conductor

\[
\oint \vec{E} \cdot \vec{dA} = 0 \quad \text{because } E=0 \text{ inside a conductor}
\]

\[
\oint \vec{E} \cdot \vec{dA} = 4\pi(Q + Q_{\text{inside}}) \quad \text{Gauss's law}
\]

$\Rightarrow Q_{\text{inside}} = -Q \Rightarrow Q_{\text{outside}} = -Q_{\text{inside}} = Q$ (conductor is overall neutral)
Corollary 3

The induced charge density on the surface of a conductor caused by a charge $Q$ inside it is $\sigma_{\text{induced}} = E_{\text{surface}}/4\pi$

Why?
- For surface charge layer, Gauss tells us that $\Delta E = 4\pi \sigma$
- Since $E_{\text{inside}} = 0 \Rightarrow E_{\text{surface}} = 4\pi \sigma_{\text{induced}}$

Uniqueness theorem

Given the charge density $\rho(x,y,z)$ in a region and the value of the electrostatic potential $\phi(x,y,z)$ on the boundaries, there is only one function $\phi(x,y,z)$ which describes the potential in that region.

Prove:
- Assume there are 2 solutions: $\phi_1$ and $\phi_2$: they will satisfy Poisson:
  $$\nabla^2 \phi_1(\vec{r}) = 4\pi \rho(\vec{r})$$
  $$\nabla^2 \phi_2(\vec{r}) = 4\pi \rho(\vec{r})$$
- Both $\phi_1$ and $\phi_2$ satisfy boundary conditions: on the boundary, $\phi_1 = \phi_2 = \phi$
- Superposition: any combination of $\phi_1$ and $\phi_2$ will be solution, including $\phi_3 = \phi_2 - \phi_1$:
  $$\nabla^2 \phi_3(\vec{r}) = \nabla^2 \phi_2(\vec{r}) - \nabla^2 \phi_1(\vec{r}) = 4\pi \rho(\vec{r}) - 4\pi \rho(\vec{r}) = 0$$
- $\phi_3$ satisfies Laplace: no local maxima or minima inside the boundaries
- On the boundaries $\phi_3 = 0 \Rightarrow \phi_3 = 0$ everywhere inside region
  $\Rightarrow \phi_1 = \phi_2$ everywhere inside region

Why do I care?
A solution is THE solution!
Uniqueness theorem: application 1

- A hollow conductor is charged until its external surface reaches a potential (relative to infinity) $\phi = \phi_0$.
  What is the potential inside the cavity?

Solution

$\phi = \phi_0$ everywhere inside the conductor’s surface, including the cavity.

Why? $\phi = \phi_0$ satisfies boundary conditions and Laplace equation
  $\Rightarrow$ The uniqueness theorem tells me that is THE solution.

Uniqueness theorem: application 2

- Two concentric thin conductive spherical shells or radii $R_1$ and $R_2$ carry charges $Q_1$ and $Q_2$ respectively.
  - What is the potential of the outer sphere? ($\phi_{\text{infinity}} = 0$)
  - What is the potential on the inner sphere?
  - What at $r=0$?

Solution

- Outer sphere: $\phi_1 = (Q_1 + Q_2)/R_1$
  - Inner sphere $\phi_i - \phi = -\int E \cdot ds = -\int \frac{Q_i}{r} dr = \frac{Q_i}{R_i} \cdot \frac{Q_i}{R_2}$
    $\Rightarrow \phi_2 = \frac{Q_2}{R_2} + \frac{Q_1}{R_1}$ Because of uniqueness: $\phi(r) = \phi_i \forall r < R_2$
Next time...

- More on Conductors in Electrostatics
- Capacitors

- NB: All these topics are included in Quiz 1
  scheduled for Tue October 5: just 2 weeks from now!!!

- Reminders:
  - Lab 1 is scheduled for Tomorrow 5-8 pm
  - Pset 2 is due THIS Fri Sep 24