Electric current $I$

- Consider a region in which there is a flow of charges:
  - E.g. cylindrical conductor

- We define a current:
  the charge/unit time flowing through a certain surface

\[ I = \frac{dQ}{dt} \]

- Units:
  - cgs: esu/s
  - SI: C/s=ampere (A)
  - Conversion: 1 A = 2.998 x 10^9 esu/s
Current density $J$

- Number density: $n = \# \text{charges} / \text{unit volume}$
- Velocity of each charge: $u$

![Diagram of current flow through an area](image)

- Current flowing through area $A$: $I = \Delta Q / \Delta t$
  - Where $\Delta Q = q \times \text{number of charges in the prism}$
  \[ I = \frac{\Delta Q}{\Delta t} = \frac{q \Delta N}{\Delta t} = \frac{q n V_{\text{prism}}}{\Delta t} = \frac{q n \cos \theta u \Delta t}{\Delta t} = q n u \cdot \hat{A} = \vec{J} \cdot \hat{A} \]
- Where we defined the current density $J$ as: $\vec{J} \equiv q n u \equiv \rho \vec{u}$

More realistic case...

- We made a number of unrealistic assumptions:
  - only 1 kind of charge carriers: we could have several, e.g.: + and - ions
  - $u$ assumed to be the same for all particles: unrealistic!
  - regular surface with $J$ constant on it
- Multiple charge carriers: $\vec{J} \equiv \sum_k q_k n_k \vec{u}_k = \sum_k \rho_k \vec{u}_k$
  - E.g.: solution with different kind of ions
  - NB: + ion with velocity $u_k$ is equivalent to - ion with velocity $-u_k$
- Velocity:
  - Not all charges have the same velocity $\rightarrow$ average velocity $\langle \vec{u} \rangle = \frac{1}{N} \sum_i \langle \vec{u}_i \rangle$, $\vec{J} \equiv \sum_k q_k n_k \langle \vec{u}_k \rangle = \sum_k \rho_k \langle \vec{u}_k \rangle$
- Arbitrary surface $S$, arbitrary $\vec{J}$: $I = \int_S \vec{J} \cdot d\vec{A}$
Non standard currents

- We usually think of currents as electrons moving inside a conductor
  - This is only one of the many examples!

- Other kinds of currents
  - Ions in solution such as Salt (NaCl) in water (Demo F5)

![Diagram of Na+ and Cl- ions]

The continuity equation

- A current $I$ flows through the closed surface $S$:
  - Some charge enters
  - Some charge exits

- What happens to the charge after it enters?
  - Piles up inside
  - Leaves the surface $\int_S \vec{J} \cdot d\vec{A} = -\frac{\partial Q_{\text{inside}}}{\partial t}$
    - NB: - because dA points outside the surface

- Apply Gauss's theorem and obtain continuity equation:

\[
\begin{align*}
\int_S \vec{J} \cdot d\vec{A} &= \int_V \nabla \cdot \vec{J} dV \\
-\frac{\partial}{\partial t} Q_{\text{inside}} &= -\frac{\partial}{\partial t} \int_V \rho dV \\
\Rightarrow \int_V \left( \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \rho \right) dV &= 0 \Rightarrow \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \rho = 0
\end{align*}
\]
Thoughts on continuity equation

- **Continuity equation:**
  \[ \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \rho = 0 \]

- **What does it teach us?**
  - Conservation of electric charges in presence of currents
  - For steady currents:
    - no accumulation of charges inside the surface: \( \frac{d\rho}{dt}=0 \)

  \[ \nabla \cdot \mathbf{J} = 0 \]

Microscopic Ohm’s law

- Electric fields cause charges to move
- Experimentally, it was observed by Ohm that
  \[ \mathbf{J} = \sigma \mathbf{E} \]

- Microscopic version of Ohm’s law:
  - It reflects the proportionality between \( \mathbf{E} \) and \( \mathbf{J} \) in each point
  - Proportionality constant: **conductivity** \( \sigma \)
Macroscopic Ohm’s law

- Current is flowing in a uniform material of length $L$ in uniform electric field $\mathbf{E} \parallel L$

- Potential difference between two ends: $V = EL$
- Ohm’s law $J = \sigma E$ holds in every point:

\[
J = \sigma E \Rightarrow \frac{I}{A} = \sigma \frac{V}{L} \Rightarrow \frac{V}{I} = R
\]

where

\[
R = \frac{L}{\sigma A}
\]

Resistance $R$

- Proportionality constant between $V$ and $R$ in Ohm’s law

\[
R = \frac{L}{\sigma A} = \frac{\rho L}{A}
\]

- Units: $[V] = [R][I]$
  - SI: Ohm ($\Omega$) = V/A
  - cgs: s/cm

- Dependence on the geometry:
  - Inversely proportional to $A$ and proportional to $L$

- Dependence on the property of the material:
  - Inversely proportional to conductivity
Resistivity

- Resistivity $\rho = 1/\sigma$
  - Describes how fast electrons can travel in the material
  - Units: in SI: $\Omega \cdot m$; in cgs: $s$

<table>
<thead>
<tr>
<th>Material</th>
<th>Resistivity (Ω·m)</th>
<th>Resistivity (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>$1.6 \times 10^{-8}$</td>
<td>$1.8 \times 10^{-17}$</td>
</tr>
<tr>
<td>Copper</td>
<td>$1.7 \times 10^{-8}$</td>
<td>$1.9 \times 10^{-17}$</td>
</tr>
<tr>
<td>Gold</td>
<td>$2.4 \times 10^{-8}$</td>
<td>$2.6 \times 10^{-17}$</td>
</tr>
<tr>
<td>Iron</td>
<td>$1.0 \times 10^{-7}$</td>
<td>$1.1 \times 10^{-16}$</td>
</tr>
<tr>
<td>Sea water</td>
<td>0.2</td>
<td>$2.2 \times 10^{-10}$</td>
</tr>
<tr>
<td>Polyethylene</td>
<td>$2.0 \times 10^{11}$</td>
<td>220</td>
</tr>
<tr>
<td>Glass</td>
<td>$\sim 10^{12}$</td>
<td>$\sim 10^3$</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>$7.5 \times 10^{17}$</td>
<td>$8.3 \times 10^8$</td>
</tr>
</tbody>
</table>

- Depends on chemistry of material, temperature,...
  - Demos F1 and F4

Resistivity vs. Temperature

- Does resistivity depend on T?
  - Demos F1 and F4

- Why?
  - Room temperature:
    - $\rho$ depends upon collisional processes
    - when $T$ increases $\rightarrow$ more collisions $\rightarrow$ $\rho$ increases
  - Very low temperature:
    - Mean free path dominated by impurities or defects in the material $\rightarrow$ $\sim$ constant with temperature.
    - With sufficient purity, some metals become superconductors
Application:

Resistivity of a spherical shell

- 2 concentric spheres; material in between has resistivity \( \rho \)
- Difference in potential \( V \rightarrow \) current
  - \( \phi_{\text{inner}} = V; \phi_{\text{outer}} = 0 \)

Q: what is the resistance \( R \)?

- Microscopic Ohm will hold: \( J = \sigma E \)
- Spherical symmetry \( \rightarrow \) spherical potential:
  \[ \phi(r) = A + \frac{B}{r} \]
- Boundary conditions: \( \phi(a) = V \) and \( \phi(b) = 0 \)
  \[ \phi(r) = V \left( \frac{ab}{b-a} \frac{1}{r} - \frac{a}{b-a} \right) \]

\[ E = -\nabla(\phi): \quad \vec{E}(r) = V \frac{ab}{b-a} \frac{1}{r^2} \Rightarrow J = \sigma V \frac{ab}{b-a} \frac{1}{r^2} \]

\[ I = \int_{\text{Sphere}} \vec{J} \cdot d\vec{A} = \int_{\text{Sphere}} \vec{J} \cdot \vec{A} = 4\pi \sigma V \frac{ab}{b-a} \Rightarrow R = \frac{V}{I} = \frac{V}{4\pi \sigma V} \frac{ab}{b-a} = \frac{b-a}{4\pi \sigma ab} \]

What if \( \sigma \) is not constant?

- Cylindrical wire made of 2 conductors with conductivity \( \sigma_1 \) and \( \sigma_2 \)

  \[ I \]

  \[ \sigma_1 \quad \sigma_2 \]

- What is the consequence?
  - Current flowing must be the same in the whole cylinder
    \[ I = A\sigma_1 E_1 = A\sigma_2 E_2 \]
  - Electric fields are different in the 2 regions
  - \( E \) discontinuous \( \rightarrow \) surface layer \( \sigma_q \) at the boundary
    \[ \sigma_q = \frac{E_{\text{surface}}}{4\pi} = \frac{E_2 - E_1}{4\pi} = \frac{I(\rho_2 - \rho_1)}{4\pi A} \]

When conductivity changes there is the possibility that some charge accumulates somewhere. This is necessary to maintain steady flow.
Thoughts on Ohm’s law

- Ohm’s law in microscopic formulation: \( \vec{J} = \sigma \vec{E} \)
  - In plain English:
    - A constant electric field creates a steady current: \( \vec{E} \propto \vec{v} \)
    - Does this make sense? \( \vec{F} = m\vec{a} \Rightarrow \vec{E} \propto \vec{a} \)
- Charges are moving in an effectively viscous medium
  - As sky diver in free fall: first accelerate, then reach constant \( v \)
  - Why? Charges are accelerated by \( E \) but then bump into nuclei and are scattered \( \rightarrow \) the average behavior is a uniform drift

\[ \begin{align*}
E
\end{align*} \]

Motion of electrons in conductor

- \( N \) electrons are moving in a material immersed in \( \vec{E} \)
  - Two components contribute to the momentum:
    - Random collision velocity \( u_0 \): \( \vec{p}_{\text{Random}} = m\vec{u}_0 \)
    - Impulse due to electric field: \( \vec{p}_E = q\vec{E}t \)
  - The average momentum is:
    \[ \langle p \rangle = m\langle u \rangle = \frac{1}{N} \sum_{i=1}^{N} (m\vec{u}_i + q\vec{E}t_i) = m \frac{1}{N} \sum_{i=1}^{N} \vec{u}_i + q\vec{E} \frac{1}{N} \sum_{i=1}^{N} t_i \]
  - For large \( N \): \( \sum_{i=1}^{N} \vec{u}_i \rightarrow 0 \) \( \rightarrow \) \( m\langle u \rangle = q\vec{E} \frac{1}{N} \sum_{i=1}^{N} t_i = q\vec{E}\tau \)
    - Where \( \tau = \frac{1}{N} \sum_{i=1}^{N} t_i \) is the average time between 2 collisions
      - Property of the material
Conductivity

- From this derivation we can read off the conductivity

\[
\begin{align*}
\bar{J} &= nq \langle \bar{u} \rangle \\
m \langle \bar{u} \rangle &= q\bar{E}\tau
\end{align*}
\Rightarrow \quad \bar{J} = nq \frac{q\bar{E}\tau}{m} = \sigma \bar{E} \quad \Rightarrow \quad \sigma = \frac{nq^2\tau}{m}
\]

- For multiple carriers:

\[
\sigma = \sum_{i=1}^{N} n_i q_i^2 \tau_i \frac{1}{m_i}
\]