Problem 1 (30 points) Entropy of a Surface Film

\( \gamma \) and \( C_A \) are given in terms of \( T \) and \( A \) so it is reasonable to choose \( T \) and \( A \) as the variables in which to expand the entropy.

\[
dS = \left( \frac{\partial S}{\partial T} \right)_A dT + \left( \frac{\partial S}{\partial A} \right)_T dA
\]

\[
C_A \equiv \left. \frac{dQ}{dt} \right|_A = T \left( \frac{\partial S}{\partial T} \right)_A \Rightarrow \left( \frac{\partial S}{\partial T} \right)_A = C_A = \frac{Nk_B}{T} + \frac{Nk_BT}{T_0^2}
\]

To find \((\partial S/\partial A)_T\) use a Maxwell Relation. You may either use the magic square or derive the required relation as follows.

\[
F \equiv U - TS
\]

\[
dF = -SdT + \gamma dA
\]

cross derivatives of the prefactors of the differentials are equal

\[
\left( \frac{\partial S}{\partial A} \right)_T = -\left( \frac{\partial \gamma}{\partial T} \right)_A = \frac{Nk_B}{A - bN}
\]
Substituting in these results gives

\[
dS = \left( \frac{N_k B}{T} + \frac{N_k B T}{T_0^2} \right) dT + \left( \frac{N_k B}{A - bN} \right) dA
\]

\[
S = N_k B \ln T + \frac{1}{2} N_k B \left( \frac{T}{T_0} \right)^2 + f(A)
\]

\[
\left( \frac{\partial S}{\partial A} \right)_T = f'(A) = \frac{N_k B}{A - bN} \Rightarrow f(A) = N_k B \ln(A - bN) + c
\]

\[
S(T, A) = N_k B \ln T + \frac{1}{2} N_k B \left( \frac{T}{T_0} \right)^2 + N_k B \ln(A - bN) + c
\]

[Note: One can make the arguments of the logs dimensionless by distributing part of the additive constant \( c \) among the various other terms.]

\[
S(T, A) = N_k B \ln(T/T_1) + \frac{1}{2} N_k B \left( \frac{T}{T_0} \right)^2 + N_k B \ln((A - bN)/A_1) + c'
\]
Problem 2 (40 points) Crystal Field Splitting

a) 
\[ Z_1 = 1 + 2 \exp[-\Delta/k_B T] \]
\[ < \epsilon > = \sum_{\text{states}} \epsilon_{\text{state}} p(\text{state}) = \frac{2\Delta \exp[-\Delta/k_B T]}{1 + 2 \exp[-\Delta/k_B T]} = 2\Delta \frac{1}{\exp[\Delta/k_B T] + 2} \]
\[ U(T, N) = N < \epsilon > = \frac{2\Delta N}{\exp[\Delta/k_B T] + 2} \]

b) At \( T = 0 \) only the non-degenerate ground state is occupied. \( S(T = 0, N) = k_B N \ln(1) = 0. \)

As \( T \to \infty \), all three states are equally probable. \( S(T, N) \to k_B N \ln(3). \)

c)
\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V = 2\Delta N \frac{d}{dT} \left( \frac{1}{\exp[\Delta/k_B T] + 2} \right) \]
\[ = 2\Delta N \left( \frac{\Delta/k_B T^2}{\exp[\Delta/k_B T] + 2} \right) \frac{\exp[\Delta/k_B T]}{(\exp[\Delta/k_B T] + 2)^2} \]
\[ = 2Nk_B \left( \frac{\Delta}{k_B T} \right)^2 \frac{\exp[\Delta/k_B T]}{(\exp[\Delta/k_B T] + 2)^2} \]

d)
\[ F(T, N) = -k_B T \ln Z = -Nk_B T \ln Z_1 = -Nk_B T \ln(1 + 2 \exp[-\Delta/k_B T]) \]
\[ P(T, N) = -\left( \frac{\partial F}{\partial V} \right)_T = -\left( \frac{\partial F}{\partial \Delta} \right)_T \frac{d\Delta}{dV} = \gamma \left( \frac{\Delta}{V} \right) \left( \frac{\partial F}{\partial \Delta} \right)_T \]
\[ = -Nk_B T \gamma \left( \frac{\Delta}{V} \right) \frac{1}{Z_1} \left( \frac{-2}{k_B T} \right) \exp[-\Delta/k_B T] \]
\[ = 2N \gamma \left( \frac{\Delta}{V} \right) \frac{\exp[-\Delta/k_B T]}{1 + 2 \exp[-\Delta/k_B T]} \]
\[ = \left( \frac{\gamma}{V} \right) 2N \Delta \frac{1}{\exp[\Delta/k_B T] + 2} = \gamma \frac{U}{V} \]
Problem 3 (30 points) Heating a Shell

a) For the shell,

\[
P_{\text{in}} = 4\pi r^2 \sigma T_H^4
\]

\[
P_{\text{out}} = 4\pi R^2 \sigma T_S^4
\]

\[P_{\text{out}} = P_{\text{in}} \Rightarrow r^2 T_H^4 = R^2 T_S^4 \rightarrow T_S = T_H \sqrt{\frac{r}{R}}\]

b) \([e(\omega, T)]_{\text{heater}} = (1) \left( \frac{1}{4} \right) c u(\omega, T_H)\]

\[P_{\text{in}} = (4\pi r^2) \left( \frac{c}{4} \right) \int_0^{\omega_0} \left( \frac{k_B T_H}{\pi^2 c^3} \right) \omega^2 d\omega\]

\[= \frac{r^2 k_B T_H}{\pi c^2} \int_0^{\omega_0} \omega^2 d\omega = \frac{k_B \omega_0^3}{3\pi c^2} r^2 T_H\]

Note that the power is coming from the central object (not from the shell) and from its surface (not volume). Thus this result is proportional to \(r^2\).

c) \([e(\omega, T)]_{\text{shell}} = \alpha(\omega) \left( \frac{1}{4} \right) c u(\omega, T_S)\]

\[P_{\text{out}} = (4\pi R^2) \left( \frac{c}{4} \right) \int_0^{\omega_0} \left( \frac{k_B T_S}{\pi^2 c^3} \right) \omega^2 d\omega = \frac{k_B \omega_0^3}{3\pi c^2} R^2 T_S\]

\[P_{\text{out}} = P_{\text{in}} \Rightarrow T_S = T_H \left( \frac{r}{R} \right)^2\]

This is an example of a poor absorber being a poor emitter (Kirchoff’s law, on the information sheet). The shell does not absorb beyond \(\omega_0\), thus it does not radiate beyond \(\omega_0\).