(17) Canonical Transforms

To motivate our next theoretical step, canonical transformations, let’s remind ourselves how we use Lagrangians and Hamiltonians to solve mechanics problems. I’ll use the simple pendulum as a concrete example.

How to do mechanics, step by step

1) Write $T$ and $U$ in Cartesian coordinates

$$T(\dot{\mathbf{r}}) = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) \quad U(\mathbf{r}) = mgy$$

2) Write transformation to generalized coordinates

$\mathbf{r}(q)$ e.g. $x = R \sin \phi \quad y = -R \cos \phi$

$\dot{\mathbf{r}}(q, \dot{q})$ e.g. $\dot{x} = R \cos \phi \dot{\phi} \quad \dot{y} = R \sin \phi \dot{\phi}$

3) Write $T(q, \dot{q})$ and $U(q)$

$$T(\phi, \dot{\phi}) = \frac{1}{2} m R^2 \dot{\phi}^2 \quad U(\phi) = -mgR \cos \phi$$

4) Compute generalized momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{with} \quad L = T - U \quad \text{e.g.} \quad p_\phi = mR^2 \dot{\phi}$$

From here, you can continue on the Lagrangian path and...

Lagrangian

5) Compute $F_i = \frac{\partial L}{\partial \dot{q}_i}$ e.g. $F_\phi = mgR \sin \phi$

6) Find equations of motion with $\dot{p}_i = F_i$ e.g. $\ddot{\phi} = \frac{g}{R} \sin \phi$

or you can use these generalized momenta in the Hamiltonian
5) Write \( T(p, q) \) e.g. \( T(p_\phi, \phi) = \frac{p_\phi^2}{2mR^2} \)

6) Find equations of motion with \( H = T + U \) and

\[
\dot{q} = \frac{\partial H}{\partial p} , \quad \dot{p} = -\frac{\partial H}{\partial q}
\]
e.g. \( \dot{\phi} = \frac{p_\phi}{mR^2} , \quad \dot{p}_\phi = mgR \sin \phi \)

It may seem strange that we need to go through the Lagrangian to find the momenta used in the Hamiltonian, but this just highlights a difference between these two approaches.

The Lagrangian is based on a choice of generalized coordinates. Any choice will do, and the momenta are a result of that choice.

\[
L'\left(Q, \dot{Q}\right) = L\left(q(Q), \dot{q}(Q, \dot{Q})\right)
\]

for any \( Q(q) \) transform (invertable, differentiable,...) e.g. from Cartesian to polar in 2D

\[
Q(q) \Rightarrow r = \sqrt{x^2 + y^2} , \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)
\]
\[
\Rightarrow L'\left(r, \phi, \dot{r}, \dot{\phi}\right) = L\left(x(r, \phi), y(r, \phi), \dot{x}(\ldots), \dot{y}(\ldots)\right)
\]

Momenta result from our choice of coordinates

\[
p_i = \frac{\partial L}{\partial \dot{q}_i} , \quad P_i = \frac{\partial L'}{\partial \dot{Q}_i}
\]

and you plug this into E-L and get the EoM. Easy.

The Hamiltonian, on the other hand, offers no clear connection between \( p \) and \( q \). You have a lot more freedom in that \( q \) need not even be a spatial coordinate, nor \( p \) related to the velocity of anything. But, if you forego that freedom and \( q \) is a generalized spatial coordinate, then...
Given some generalized coordinates \( q_i \),
the momenta \( p_i = \frac{\partial L}{\partial \dot{q}_i} \) are those required for \( H(p, q) \)

Does this mean we need to construct \( L(q, \dot{q}) \) every time we want to change coordinates with \( H \)?

**No! There are 3 other ways...**
In each case we start with steps 1 and 2.
For the first path, we take step 3 and note that momenta are usually easy to guess (e.g. \( \vec{p} = m \vec{\dot{r}} \)).

**Path 1: “guess and check”**
Guess your momenta \( P(p, q) \)

\[ \text{e.g. } p_\phi = mR^2 \dot{\phi} = L_z = xp_y - yp_x \]

and check the Poisson Brackets (necessary and sufficient)

\[[Q_j, Q_k]_{pq} = 0,\ [P_j, P_k]_{pq} = 0,\ [P_j, Q_k]_{pq} = \delta_{jk} \]

\[\text{e.g. } [\phi, \phi] = [\tan^{-1}\left(\frac{-x}{y}\right), \tan^{-1}\left(\frac{-x}{y}\right)] = 0 \]

\[[p_\phi, p_\phi] = 0 \ ( [f, f] = 0 \text{ for any } f) \]

\[[p_\phi, \phi] = [xp_y - yp_x, \tan^{-1}\left(\frac{x}{y}\right)] \]

\[= x [p_y, \tan^{-1}\left(\frac{-x}{y}\right)] - y [p_x, \tan^{-1}\left(\frac{-x}{y}\right)] \]

\[= x \frac{\partial}{\partial y} \tan^{-1}\left(\frac{-x}{y}\right) - y \frac{\partial}{\partial x} \tan^{-1}\left(\frac{-x}{y}\right) \]

\[= \frac{x^2}{R^2} + \frac{y^2}{R^2} = 1 \]
Of course, we only have one generalized coordinate, \( \phi \), in this example. In general, you will have \( \frac{3}{2}n(n-1) \) non-trivial PB to compute which give zero, and \( n \) of them which give 1, to perform this check. If \( n > 2 \), you'll need a computer or a free weekend.

Result:

\[
H = \frac{p_x^2 + p_y^2}{2m} + mg y \Rightarrow H' = \frac{p_\phi^2}{2mR^2} - mgR \cos \phi
\]

Paths 2 and 3 are similar and require some back story. Remember that curious fact about Lagrangians that adding the total time derivative of a function doesn’t change the equation of motion? (LL eq. 2.8) I promised we would get back to that and here we are.

Recall:

\[
L' = L + \frac{d}{dt}f(q, t) \Rightarrow \text{same EoM}
\]

and \( L(q, \dot{q}, t) = L'(Q, \dot{Q}, t) \Rightarrow \text{same E-L} \)

\[
\Rightarrow p\dot{q} - H = P\dot{Q} - H' + \frac{d}{dt}F(q, Q, p, P, t)
\]

If we limit \( F \) to be a function of one old variable (\( p \) or \( q \)) and one new variable (\( P \) or \( Q \)) it is called a “generating function”. There are 4 ways we can do this, each with its own implications for the transformation (from \( p, q \) to \( P, Q \)) that results. The general rules are

for \( F_1(q, Q) \)

\[
p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}
\]

for \( F_2(q, P) \)

\[
p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}
\]
for $F_3(p, Q)$  
$q_i = \frac{-\partial F_3}{\partial p_i}, \quad P_i = \frac{-\partial F_3}{\partial Q_i}$

for $F_4(p, P)$  
$q_i = \frac{-\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}$

So, if you want to make a coordinate transform with a Hamiltonian, you either do it through the Lagrangian, you guess and check with Poisson Brackets, or you find a generating function.

Let’s do this for our pendulum example. Given a coordinate transform from old to new, we use $F_2$

Given $\bar{Q}(\bar{q})$, use $F_2(q, P) = \bar{Q}(\bar{q}) \cdot \bar{P} = \sum Q_i(\bar{q}) P_i$

$\Rightarrow Q_i = \frac{\partial F_2}{\partial P_i} = Q_i(q), \quad p_i = \frac{\partial F_2}{\partial q_i}$

For pendulum

$Q(\bar{q}) \Rightarrow \phi(x, y) = \tan^{-1}\left(\frac{-x}{y}\right)$

$F_2 = \bar{Q}(\bar{q}) \cdot \bar{P} \Rightarrow F_2(x, y, p_\phi) = \tan^{-1}\left(\frac{-x}{y}\right) p_\phi$

This generating function is constructed to make $\frac{\partial F_2}{\partial P_i}$ trivially give us the desired $Q_i$ (point transform). The second differential gives us the new momenta.

\[ p_x = \frac{\partial F_2}{\partial x} = p_\phi \frac{\partial}{\partial x} \tan^{-1}\left(\frac{-x}{y}\right) = p_\phi \left(\frac{-y}{R^2}\right) = p_\phi \frac{\cos \phi}{R} \]

\[ p_y = \frac{\partial F_2}{\partial y} = p_\phi \frac{\partial}{\partial y} \tan^{-1}\left(\frac{-x}{y}\right) = p_\phi \left(\frac{x}{R^2}\right) = p_\phi \frac{\sin \phi}{R} \]
Since I only have one new momenta and two old, this is over constrained, and both give the same answer. The cartesian momenta are

\[
\begin{align*}
    p_x &= m \dot{x} = mR \cos \phi \dot{\phi} \Rightarrow p_\phi = mR^2 \dot{\phi} \\
    p_y &= m \dot{y} = mR \sin \phi \dot{\phi}
\end{align*}
\]

where we inverted either expression to get \( p_\phi \). This matches our guess, so we have \( H(p,q) \).

We can also use the \( F_3 \) generator function in a similar way. Again, we trivially recover our point transform, with the first differential equation,

Given \( \tilde{q}(\tilde{Q}) \), use \( F_3(\tilde{p}, \tilde{Q}) = -\tilde{q}(\tilde{Q}) \cdot \tilde{p} \)

\[
\Rightarrow q_i = -\frac{\partial F_3}{\partial p_i} = q \left( \tilde{Q} \right) , \quad P_i = -\frac{\partial F_3}{\partial Q_i}
\]

and the second gives us the new momenta \( P \).

For pendulum

\[
\begin{align*}
    \tilde{q}(\tilde{Q}) \to x &= R \sin \phi , \quad y = -R \cos \phi \\
    F_3(p_x, p_y, \phi) &= -R \sin \phi p_x + R \cos \phi p_y
\end{align*}
\]

\[
\begin{align*}
    p_\phi &= -\frac{\partial F_3}{\partial \dot{\phi}} = R \left( \cos \phi p_x + \sin \phi p_y \right) \\
    &= \left( \cos \phi \left( m \dot{x} \right) + \sin \phi \left( m \dot{y} \right) \right) \\
    &= mR \left( \cos \phi \left( R \cos \phi \dot{\phi} \right) + \sin \phi \left( R \sin \phi \dot{\phi} \right) \right) \\
    &= mR^2 \dot{\phi}
\end{align*}
\]

For our example, in which the coordinate transform is most easily expressed as \( \tilde{q}(\tilde{Q}) \), this path through \( F_3 \) is the most direct way to go from step 2 to step 5.
without passing through $L$ (at the price of needing to invert $P_i = f(\vec{r}, \vec{q})$).

Of course, we have explored only a very limited range of generator functions. These needn’t result in point transforms: the Hamiltonian is not limited like the Lagrangian to point transforms $Q(q) \Rightarrow \dot{Q}(q, \dot{q})$. Rather, you can have $Q(p,q)$ and $P(p,q)$.

For instance, let’s try this...

**Transform $H$ from $\phi, p_\phi$**

$$H(p_\phi, \phi) = \frac{p_\phi^2}{2mR^2} - mgR \cos \phi$$

$$\approx \frac{p_\phi^2}{2I} + \frac{k}{2} \phi^2 + \text{const} \quad \text{for} \quad \phi \ll 1$$

with $I = mR^2$, $k = mgR = \frac{gI}{R} = I\omega^2$.

Dropping the constant gives us the Hamiltonian of a simple harmonic oscillator with frequency $\omega = \sqrt{\frac{g}{R}}$.

**try** $F_1(\phi, \theta) = \frac{I\omega \phi^2}{2\tan \theta}$ with $\omega = \sqrt{\frac{g}{R}}$

$$p_\phi = \frac{\partial F_1}{\partial \phi} = \frac{I\omega \phi}{\tan \theta}$$

$$\Rightarrow \tan \theta = \frac{I\omega \phi}{p_\phi}$$

$$p_\theta = -\frac{\partial F_1}{\partial \theta} = \frac{I\omega \phi^2}{2} \frac{\partial}{\partial \theta} \frac{-1}{\tan \theta} = \frac{I\omega \phi^2}{2 \sin^2 \theta}$$

Now we need to write $H(p_\theta, \theta)$ based on $H(p_\phi, \phi)$, just like we got $H(p_\phi, \phi)$ from $H$ in cartesian coordinates.
Find \( H(p_\theta, \theta) \)

\[
H(p_\theta, \theta) = \frac{p_\phi^2}{2I} + \frac{I\omega^2 \phi^2}{2} = \frac{I\omega^2 \phi^2}{2\tan^2 \theta} + \frac{I\omega^2 \phi^2}{2}
\]

use \( \frac{1}{\tan^2 \theta} + 1 = \frac{1}{\sin^2 \theta} \Rightarrow H = \omega p_\theta \)

Now that is a simple Hamiltonian!

EoM for \( \theta \)

\[
\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \omega , \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0
\]

\( \theta = \omega t + \theta_0 , \quad p_\theta = \text{const} \)

What does this mean physically? Let’s return to our angular coordinate \( \phi \);

\[
\phi = \sqrt{\frac{2p_\theta}{I\omega}} \sin(\omega t + \theta_0)
\]

\[
p_\phi = \sqrt{2p_\theta I\omega} \cos(\omega t + \theta_0)
\]

SHO:

\( E = \frac{1}{2} I\omega^2 A^2 \Rightarrow p_\theta = \frac{E}{\omega} \)

so \( \theta \) is the phase of the oscillator, and \( p_\theta \) is related to the energy of the oscillation. (Note that \( H = E \) as expected.)

So this generator function moved us into a “coordinate” system where our “momentum” was actually energy (a constant) and “position” was actually the phase of the harmonic oscillator solution!

This would not work with a Lagrangian!
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January IAP 2017

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