THE SCHWARZSCHILD METRIC

\[ ds^2 = -c^2d\tau^2 = - \left( 1 - \frac{2GM}{rc^2} \right) c^2dt^2 \left( 1 - \frac{2GM}{rc^2} \right)^{-1} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \]

\[ R_S = \frac{2GM}{c^2} \]

\[
\frac{d}{d\tau} \left[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] = \frac{1}{2} \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau}. 
\]

Use indices \( \mu, \nu, \) etc., which are summed from 0 to 3, where \( x^0 \equiv t. \)

Use \( \tau \) to parameterize the path, where \( \tau = \) proper time measured along the path.

RADIAL GEODESICS

\[ \frac{d}{d\tau} \left[ g_{rr} \frac{dr}{d\tau} \right] = \frac{1}{2} \partial_r g_{rr} \left( \frac{dr}{d\tau} \right)^2 + \frac{1}{2} \partial_r g_{tt} \left( \frac{dt}{d\tau} \right)^2. \]

which simplifies to

\[ \frac{d^2r}{d\tau^2} = -\frac{GM}{r^2}. \]

It looks like Newton, but \( r \) is not really the distance from the origin, and \( \tau \) is the proper time measured along the trajectory.
The proper time $\tau$ needed to reach radial variable $r$ is

$$\tau(r) = \sqrt{\frac{r_0}{2GM}} \left\{ r_0 \tan^{-1}\left( \frac{r_0 - r}{r} \right) + \sqrt{r(r_0 - r)} \right\}.$$ 

The infalling object will be ripped apart by the singularity at $r = 0$ in a finite amount of the object’s proper time.

But from the outside, it will take an infinite amount of coordinate time $t$ before the object reaches the horizon.