3: GENERAL ASPECTS OF QUANTUM ELECTRODYNAMICS

3.1: RENORMALIZED LAGRANGIAN

Consider the Lagrangian of quantum electrodynamics in terms of the bare quantities:

\( \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^B F_{\mu\nu}^B - i \bar{\psi}_B (\gamma^\mu (\partial_\mu - ie_B A_\mu^B) - m_B) \psi_B. \)  

We use the convention:

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu\nu}, \]

where, in four dimensions, \( (\eta^{\mu\nu}) = \text{diag}(-1, 1, 1, 1). \) We expect to find the mass and field renormalizations.

Note: we will omit the “B” signifying bare quantities in what follows.

We have that

\[ S_{\alpha\beta}(x) = \langle 0 | T(\psi_\alpha(x) \bar{\psi}_\beta(0)) | 0 \rangle, \]

and

\[ S_{\alpha\beta}(k) = \frac{1}{1 + \Sigma S_0}, \]
where we have omitted the spinor indices in the second and third lines: these are to be read as matrix equations. Hence, we have that

\[ S^{-1} = S_0^{-1} + \Sigma. \]  

(7)

Recall that

\[ S_0 = \frac{-1}{ik + m_B}, \quad -\Sigma = \langle \ldots \rangle \quad + \ldots, \]

(8)

and so

\[ S^{-1} = -(ik + m_B) + \Sigma, \]

(9)

and we have for the fully interacting two-point function

\[ S = \frac{-1}{ik + m_B - i\epsilon - \Sigma}. \]

(10)

Note that \( \Sigma = \Sigma(\bar{k}) \), since it can only depend on \( \bar{k} \) and \( k^2 \). Even though it is a function of matrix, we can treat it as an ordinary function. The physical mass is defined by \( \bar{k} = im \), so that

\[ -m + m_B - \Sigma(im) = 0. \]

(11)

We note that, again, \( \Sigma \) will be divergent. Near the pole, we have that

\[ S \approx \frac{-Z_2}{ik + m_B - i\epsilon \Sigma} \]

(12)

with \( Z_2^{-1} = 1 + i \frac{\partial Z_2}{\partial k} |_{k=im} \). The relations between the bare and physical quantities are given by

\[ m_B = m + \delta m, \quad \psi_B = \sqrt{Z_2} \psi \]

(13)

where \( \delta m = \Sigma(im) \) and \( \psi \) is the renormalized field.

**3.1.2: Photon self-energy**

Similarly, we have that

\[ D_{\mu\nu}(x) = \langle 0 | T(A_{\mu}^B(x)A_{\nu}^B(0)) | 0 \rangle, \]

(14)

and

\[
D_{\mu\nu}(k) = \mu \quad \ldots \quad \nu 
\quad + \quad \nu \quad \ldots 
\quad + \quad \nu \quad \ldots 
\quad + \ldots 
\]

\[ = D_0(k) + D_0(i\Pi)D_0 + D_0(i\Pi)D_0(i\Pi)D_0 + \ldots \]

\[ = D_0 \frac{1}{1 - i\Pi D_0} \]

(15)

Hence,

\[ (iD)^{-1} = (iD_0)^{-1} - \Pi. \]

(16)

Recall that

\[
id_0^{\mu\nu} = \frac{1}{k^2 - i\epsilon} \left[ \eta^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right] = \frac{1}{k^2 - i\epsilon} \left[ P_T^{\mu\nu} + \xi P_L^{\mu\nu} \right],
\]

where we have defined the transverse projector \( P_T^{\mu\nu} \equiv \eta^{\mu\nu} - k^\mu k^\nu \), and the longitudinal projector \( P_L^{\mu\nu} \equiv \frac{k^\mu k^\nu}{k^2} \). Note that it is not a coincidence that the propagator can be built from these two tensors, \( \eta^{\mu\nu} \) and \( k^\mu k^\nu \): they are the only two two-tensors allowed by symmetry. These projectors satisfy the properties

\[
P_T^{\mu\nu} P_T^{\nu\lambda} = P_T^{\mu\lambda}, \quad P_L^{\mu\nu} P_L^{\nu\lambda} = P_L^{\mu\lambda}, \quad P_T^{\mu\nu} P_L^{\nu\lambda} = 0.
\]

(17)
Note: $T$ and $L$ are just labels here, and the placing of these indices does not carry meaning. Hence, we have that
\[(iD_0)^{-1} = k^2 \left[ P^\mu_\nu + \frac{1}{\xi} P^\mu_\nu \right]. \tag{18}\]
We may also expand $\Pi^{\mu\nu}$ as
\[
\Pi^{\mu\nu} = \left. \frac{\delta L}{\delta A^\mu} \frac{\partial^2 S}{\partial A^\nu \partial \bar{\psi} \gamma^\nu} \right|_{\bar{\psi} = \psi = 0} + \left. \frac{\delta L}{\delta \bar{\psi} \gamma^\nu} \frac{\partial^2 S}{\partial A^\mu \partial \bar{\psi} \gamma^\nu} \right|_{\bar{\psi} = \psi = 0}.
\tag{19}\]
Therefore,
\[
(iD)^{-1} = P^\mu_\nu (k^2 - f_T) + \frac{1}{\xi} P^\mu_\nu \left( \frac{k^2}{\xi} - f_L \right), \tag{20}\]
and we have for the full interacting photon two-point function,
\[
D = -i \left[ P^\mu_\nu \frac{1}{k^2 - f_T} + P^\mu_\nu \frac{1}{\xi - f_L} \right]. \tag{21}\]
We observe that if $f_{T,L}(k^2 = 0) \neq 0$, a mass will be generated for the photon. Because $\Pi^{\mu\nu}$ comes from 1PI diagrams, it should not be singular at $k^2 = 0$, and so $f_L = f_T = O(k^2)$, as $k \to 0$. We will show that gauge invariance ensures that no mass is generated from the loop corrections.

3.1.3: Ward identities

Consider the path integral for the generating functional:
\[
Z \left[ J_\mu, \eta, \bar{\eta} \right] = \int \mathcal{D} A_\mu \mathcal{D} \psi \mathcal{D} \bar{\psi} e^{iS[A_\mu, \psi, \bar{\psi}]}. \tag{22}\]
where $S = S_{QED} + \int d^4x J_\mu A^\mu_\nu + \bar{\eta} \gamma^\nu \psi + \bar{\psi} B \eta$, where we note explicitly these couplings are in terms of bare quantities.
\[
\mathcal{L}_{QED} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} (\gamma^\mu D_\mu - m) \psi - \frac{1}{2\xi} (\partial_\mu A^\mu)^2. \tag{23}\]
We define the generating functional for connected diagrams, $W \left[ J_\mu, \eta, \bar{\eta} \right]$, by
\[
Z \left[ J_\mu, \eta, \bar{\eta} \right] = e^{iW[J_\mu, \eta, \bar{\eta}]} \tag{24}.
\]
For example,
\[
\langle 0 \mid T(\bar{\psi}_\alpha(x) \psi_\beta(y)) \mid 0 \rangle = i \frac{\delta^2 W \left[ J_\mu, \eta, \bar{\eta} \right]}{\delta \eta_\alpha(x) \delta \eta_\beta(y)} \bigg|_{J = \eta = \bar{\eta} = 0},
\]
\[
\langle 0 \mid T(A_\mu^B(x) A_\nu^B(y)) \mid 0 \rangle = i \frac{\delta^2 W \left[ J_\mu, \eta, \bar{\eta} \right]}{\delta J^\mu(x) \delta J^\nu(y)} \bigg|_{J = \eta = \bar{\eta} = 0}.
\]
Recall, for infinitesimal gauge transformations, $\delta A_\mu = \partial_\mu \lambda$, $\delta \psi = i\epsilon B \lambda \psi$, and $\delta \bar{\psi} = -i\epsilon B \lambda \bar{\psi}$. Consider a change of variables in the path integral:
\[
A_\mu \rightarrow A'_\mu = A_\mu + \delta A_\mu,
\psi \rightarrow \psi' = \psi + \delta \psi,
\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} + \delta \bar{\psi}.
\]
Then we have
\[
\int \mathcal{D} A'_\mu \mathcal{D} \psi' \mathcal{D} \bar{\psi}' e^{iS[A'_\mu, \psi', \bar{\psi}']} = \int \mathcal{D} A_\mu \mathcal{D} \psi \mathcal{D} \bar{\psi} e^{iS[A_\mu, \psi, \bar{\psi}]}, \tag{25}\]
as this is just of a change of the dummy integration variables. Note that the measure is unchanged by this shift:
\[
\mathcal{D} A'_\mu \mathcal{D} \psi' \mathcal{D} \bar{\psi}' = \mathcal{D} A_\mu \mathcal{D} \psi \mathcal{D} \bar{\psi}, \tag{26}\]
and the action for the two sets of variables are related by
\[
S[A'_\mu, \psi', \bar{\psi}'] = S[A'_\mu, \psi, \bar{\psi}] - \frac{1}{\xi} \int d^4x \partial_\mu A^\mu \partial^2 \lambda + \int d^4x J_\mu \partial^\mu \lambda + i e_B \lambda \bar{\eta} \psi - i e_B \lambda \bar{\psi}.
\]  
(27)

Hence, we must have
\[
\int d^4x \lambda(x) \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[A,\psi,\bar{\psi}]} \left[ -\frac{1}{\xi} \partial^2 \partial_\mu A^\mu - \partial_\mu J^\mu + i e_B (\bar{\eta} \psi - \bar{\psi} \eta) \right] = 0.
\]  
(28)

Since
\[
A_\mu(x) \sim -i \frac{\delta Z}{\delta J^\mu(x)} = Z \frac{\delta W}{\delta J^\mu(x)},
\]
\[
\psi(x) \sim -i \frac{\delta Z}{\delta \eta(x)} = Z \frac{\delta W}{\delta \eta(x)},
\]
\[
\bar{\psi}(x) \sim -i \frac{\delta Z}{\delta \bar{\eta}(x)} = Z \frac{\delta W}{\delta \bar{\eta}(x)},
\]

we have that
\[
\left. \frac{1}{\xi} \partial^2 \partial_\mu \frac{\delta^2 W}{\delta J^\mu(x) \delta J^\nu(y)} \right|_{y=\eta=0} + \partial_\nu \delta^{(4)}(x-y) = 0,
\]  
(29)

that is,
\[
\frac{i}{\xi} \partial^2 \partial_\mu D_{\mu\nu}(x-y) + \partial_\nu \delta^{(4)}(x-y) = 0,
\]  
(30)
or, written in momentum-space,
\[
-\frac{i}{\xi} k^2 k^\mu D_{\mu\nu}(k) + k_\nu = 0.
\]  
(31)

If we now write
\[
D_{\mu\nu}(k) = P^T_{\mu\nu} D_T(k^2) + P^L_{\mu\nu} D_L(k^2),
\]  
(32)

with \(k^\mu P^L_{\mu\nu} = k_\nu\), the Ward identity reduces to
\[
-\frac{i}{\xi} k^2 k_\nu D_L(k^2) + k_\nu = 0,
\]  
(33)

and so
\[
D_L(k^2) = -\frac{i \xi}{k^2},
\]  
(34)

and the longitudinal part of the two-point function is completely determined. Comparing this with (21), we find that \(f_L(k^2) = 0\), and we thus conclude that \(\Pi^{\mu\nu}\) is purely transverse. That is, from (19), we have that
\[
\Pi^{\mu\nu} = \left( \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) f_T(k^2).
\]  
(35)

For \(\Pi^{\mu\nu}(k)\) to be non-singular at \(k = 0\), we must have
\[
f_T(k^2) = k^2 \Pi(k^2),
\]  
(36)

where \(\Pi(0)\) is non-singular. Hence, for the two-point function in the interacting theory, we have
\[
D_{\mu\nu} = \frac{-i}{k^2 - i \epsilon} \left[ \frac{P^T_{\mu\nu}}{1 - \Pi(k^2)} + \xi P^L_{\mu\nu} \right].
\]  
(37)

Remarks:
1. The longitudinal part of \(D_{\mu\nu}\) does not receive any loop corrections: it is completely determined by the Ward identities. The physics should not depend on this part. For example, in the Landau gauge, \(\xi = 0\), \(D_{\mu\nu}\) is purely transverse.
2. Since $\Pi(k^2)$ is non-singular at $k^2 = 0$, the photon remains massless to all orders. There are exceptions to this: it is not true in quantum electrodynamics in $1+1$ dimensions, or in theories where an additional Higgs field is introduced.

3. The residue at the $k^2 = 0$ pole is given by $Z_3^{-1} = 1 - \Pi(0)$, and we have that

$$iD^{\mu\nu}_{\kappa\gamma} \approx \frac{Z_3}{k^2 - i\epsilon} P^{\mu\nu}_{\kappa\gamma}$$

near $k^2 = 0$. The renormalized field is given by $A^{\mu}_{\kappa\gamma} = \sqrt{Z_3}A_{\mu}$. 
