5.1: RENORMALIZATION GROUP FLOW

Consider the bare action defined at a scale $\Lambda_0$:

$$ S[\Lambda_0] = \int^{\Lambda_0} d^d x \frac{1}{2} (\partial \phi)^2 + \sum_i g_i^{(0)} O_i, \tag{1} $$

where $O_i$ is a complete set of local operators formed from $\phi$. The theory is specified by the set \( \{g_i^{(0)}\} \). As explained in the previous lecture, we can change the cutoff scale to some $\Lambda < \Lambda_0$ by integrating out the degrees of freedom in the interval $(\Lambda, \Lambda_0)$. This gives

$$ S[\Lambda] = \int^{\Lambda} d^d x \frac{1}{2} (\partial \phi)^2 + \sum_i g_i(\Lambda) O_i, \tag{2} $$

after redefining $\phi$ to absorb the field renormalization factor $Z$. This theory is specified by the set \( \{g_i(\Lambda)\} \). Similarly, at another scale $\Lambda' < \Lambda$, we obtain $S[\Lambda']$, described by \( \{g_i(\Lambda')\} \). These three actions, $S_{\Lambda_0}$, $S_\Lambda$ and $S_{\Lambda'}$, should all describe the same physics at an energy scale $E < \Lambda' < \Lambda < \Lambda_0$. The relations between them can be found by integrating out the degrees of freedom explicitly in the path integral, giving

$$ g_i(\Lambda) = g_i(g_i^{(0)}, \Lambda_0; \Lambda), $$

$$ g_i(\Lambda') = g_i(g_i^{(0)}, \Lambda_0; \Lambda') $$

$$ = g_i(g_i, \Lambda; \Lambda'). $$

This process describes the renormalization group transformations, or the renormalization group flow: transformations between couplings at different scales to ensure they describe the same low energy physics. If we consider, for

![Figure 1: The renormalization flow as the flow in the space of all possible coupling parameterizations to ensure the same low-energy physics at different scales.](image)

simplicity, the dimensionless couplings \( \{\lambda_i(\Lambda)\} \) defined by $\lambda_i \equiv g_i \Lambda^{-\delta_i}$, differentiating gives

$$ \Lambda \frac{d\lambda_i}{d\Lambda} = \beta_i(\{\lambda_j(\Lambda)\}) \tag{3} $$

where $\beta_i(\{\lambda_j(\Lambda)\}) = \frac{d}{d\ln \Lambda} \lambda_i(\{\lambda_j(\Lambda)\}, Z)|_{Z=1}$. It is important to note that the $\beta_i$ are only functions of the dimensionless coupling constants $\{\lambda_j(\Lambda)\}$: they do not depend on $\Lambda$ explicitly, as can be seen by considering
integrating out a fraction of the highest-energy modes in the path integral. The $\beta$-functions give the tangent vector of the flow, and depend only on the values of $\{\lambda_j\}$. Under a relabeling of couplings,

$$\tilde{\lambda}_i = \tilde{\lambda}_i(\{\lambda_j\}),$$

we have that

$$\tilde{\beta}_i(\{\tilde{\lambda}\}) = \sum_j \frac{d\tilde{\lambda}_i}{d\lambda_j} \beta_j(\{\lambda\}).$$

The $\beta$-functions can be computed explicitly from the path integral:

$$Z[J] = \int_{|k|<\Lambda} \mathcal{D}\phi e^{-S[\Lambda,\phi] - \int d^4x J\phi}$$

$$= \int_{|k|<\Lambda'} \mathcal{D}\phi_{\Lambda'}(k) \int_{\Lambda' <|k|<\Lambda} \mathcal{D}\tilde{\phi}(k) e^{-S[\phi_{\Lambda'} + \tilde{\phi}, \Lambda] - \int d^4x J(\phi_{\Lambda} + \tilde{\phi})}$$

$$= \int_{|k|<\Lambda} \mathcal{D}\phi_{\Lambda'}(k) e^{-S[\phi_{\Lambda'}, \Lambda'] - \int d^4x J\phi_{\Lambda'}.}$$

Now, if we let $\Lambda' \longrightarrow \Lambda - \delta \Lambda$, $S[\Lambda - \delta \Lambda] = S[\Lambda] + \delta S[\Lambda]$, we have

$$\Lambda \frac{dS_\Lambda}{d\Lambda} = F(S_\Lambda)$$

Expanding

$$S_\Lambda = \sum_i g_i O_i = \sum_i \lambda_i \Lambda^\delta O_i;$$

(6) gives us the $\beta$-functions for all couplings. As an example, let us consider the case of a free scalar field in four dimensions, with a cut-off at a scale $\Lambda$. Then, we have

$$S_\Lambda[\phi] = \int_{k<\Lambda} \frac{d^4k}{(2\pi)^4} f(k) \phi^*_\Lambda(k) \phi_\Lambda(k).$$

We expand $f(k)$ as a power series in $k$:

$$f(k) = m_0^2 + k^2 + r_4 k^4 + \ldots$$

$$= \lambda_m(\Lambda) \Lambda^2 + k^2 + \tilde{r}_4(\Lambda) k^4 + \ldots,$$

where the coefficient of $k^2$ can be chosen to one with a suitable normalization for $\phi_\Lambda$. Here, $\lambda_m(\Lambda)$, $\tilde{r}_4(\Lambda)$, . . . are dimensionless couplings: $[\phi^2] = 2$, $[\partial^2 \phi^2] = 6$, and so $\delta_m = 2$, $\delta_{\tilde{r}_4} = -2$, for example. We now let $\phi_\Lambda(k) = \phi_{\Lambda'}(k) + \hat{\phi}(k)$ with $\hat{\phi}(k)$ supported for $k \in (\Lambda', \Lambda)$ and $\phi_{\Lambda'}$ supported for $k \in (0, \Lambda')$. Then we have that

$$S_\Lambda[\phi_\Lambda] = S_\Lambda[\phi_{\Lambda'}] + S_\Lambda[\hat{\phi}] + 2 \int \frac{d^4k}{(2\pi)^4} f(k) \phi_{\Lambda'}(k) \hat{\phi}(k),$$

where the last term is zero as $\phi_{\Lambda'}$ and $\phi_k$ have disjoint support. Integrating out $\hat{\phi}$ only generates an overall constant for the path integral, and so

$$S_{\Lambda'}[\phi_{\Lambda'}] = S_\Lambda[\phi_{\Lambda'}] = \int_{k<\Lambda'} \frac{d^4k}{(2\pi)^4} f(k) \phi^*_{\Lambda'}(k) \phi_{\Lambda'}(k)$$

where $f(k)$ has not changed. That is,

$$f(k) = m_0^2 + k^2 + r_4 k^4 + \ldots$$

$$= \lambda_m(\Lambda') \Lambda'^2 + k^2 + \frac{\tilde{r}_4(\Lambda')}{\Lambda^2} k^4 + \ldots,$$
and we conclude that

\[ \lambda_m (A') = \lambda_m (A) \left( \frac{A'}{A} \right)^2 = \lambda_m (A) \left( \frac{A'}{A} \right)^{-\delta_m} \] is a relevant operator,

\[ \tilde{r}_4 (A') = \tilde{r}_4 (A) \left( \frac{A'}{A} \right)^{-2} = \tilde{r}_4 (A) \left( \frac{A'}{A} \right)^{-\delta_m} \] is an irrelevant operator.

Similarly,

\[ \beta_m = \Lambda \left. \frac{d\lambda_m (A')}{dA} \right|_{A' \to A} = -2\lambda_m = -\delta_m \lambda_m < 0, \]

\[ \beta_{r_4} = \Lambda \left. \frac{d\tilde{r}_4 (A')}{dA} \right|_{A' \to A} = 2\tilde{r}_4 = -\delta_m \tilde{r}_4 > 0. \]

We note that dimensional quantities like \( m^2 \) and \( r_4 \) do not change at all in this instance, but that the dimensionless couplings flow as they are defined with respect to the cut-off scale. This does reflect the right physics: the relative importance of each term in \( f(k) \) as we go to lower energies, or smaller \( k \). That is,

\[ \frac{m_0^2}{k^2} \] becomes larger as \( k \) becomes smaller,

\[ \frac{r_4 k^4}{k^2} \] becomes smaller as \( k \) becomes smaller.

We will now derive the full flow equation for \( S_A [\phi] \). For this purpose, we write it as

\[ S [\phi, A] = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} G^{-1}_A (k) \phi (k) \phi (-k) + S_I [\phi, A] + U(A) \] (11)

where \( U(A) \) is a cosmological constant, and the propagator \( G_A (k) \) satisfies

\[ G_A (k) = \begin{cases} \frac{1}{2\pi^2} & k \ll A, \\ 0 & k \gg A. \end{cases} \] (12)

We have that

\[ Z = \int \mathcal{D}\phi (k) e^{-S_0 [\phi, A] - S_I [\phi, A]}, \] (13)

where \( \tilde{S}_I = S_I + U \). There is now no need to impose an explicit cut-off when integrating over \( \phi (k) \). It is clearly very complicated to obtain the flow equation for \( \tilde{S}_I [\phi, A] \) by evaluating the path integral directly. We will instead require

\[ \Lambda \frac{dZ [A]}{dA} = 0, \] (14)

which is an equivalent statement. From this, we have

\[ \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left\langle \phi (-k) \phi (k) e^{-S_I} \right\rangle \Lambda \frac{dG^{-1}_A}{d\Lambda} = \left\langle \Lambda \partial_\Lambda e^{-S_I} \right\rangle. \] (15)
Here, \( \langle \ldots \rangle = \frac{1}{Z_0} \int \mathcal{D}\phi \ldots e^{-S_0} \), with \( Z_0 = \int \mathcal{D}\phi e^{-S_0} \). We would like to express the left-hand side of (15) more directly in terms of \( S_I \). For this purpose, consider
\[
0 = \int \mathcal{D}\phi \frac{\delta}{\delta \phi(k)} \left( \phi(k)e^{-S_0 - S_I} \right). \tag{16}
\]

From this, we have that
\[
(2\pi)^4 \delta^{(4)}(0) \left\langle e^{-\hat{S}_I} \right\rangle - G^{-1}_\Lambda \left\langle \phi(k)\phi(-k)e^{-\hat{S}_I} \right\rangle + \left\langle \phi(k)\frac{\delta}{\delta \phi(k)}e^{-\hat{S}_I} \right\rangle = 0. \tag{17}
\]

The last term in this equation is still complicated. Consider further
\[
0 = \int \mathcal{D}\phi \left( \frac{\delta^2}{\delta \phi(k)\delta \phi(-k)} \right) e^{-S_0 - S_I}. \tag{18}
\]

From this, we have
\[
(2\pi)^4 \delta^{(4)}(0)G^{-1}_\Lambda \left\langle e^{-\hat{S}_I} \right\rangle - (G^{-1}_\Lambda)^2 \left\langle \phi(k)\phi(-k)e^{-\hat{S}_I} \right\rangle - 2G^{-1}_\Lambda \left\langle \phi(k)\frac{\delta}{\delta \phi(k)}e^{-\hat{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta \phi(k)\delta \phi(-k)}e^{-\hat{S}_I} \right\rangle = 0. \tag{19}
\]

If we multiply 17 by \( 2G^{-1}_\Lambda \) and add the result to (19), we obtain
\[
(2\pi)^4 \delta^{(4)}(0)G^{-1}_\Lambda \left\langle e^{-\hat{S}_I} \right\rangle - (G^{-1}_\Lambda)^2 \left\langle \phi(k)\phi(-k)e^{-\hat{S}_I} \right\rangle + \left\langle \frac{\delta^2}{\delta \phi(k)\delta \phi(-k)}e^{-\hat{S}_I} \right\rangle = 0. \tag{20}
\]

Eliminating \( \left\langle \phi(k)\phi(-k)e^{-\hat{S}_I} \right\rangle \) between (15) and (20) gives
\[
\left\langle \Lambda \frac{d}{d\Lambda} e^{-\hat{S}_I - U} \right\rangle = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda}{d\Lambda} \left\langle \frac{\delta^2}{\delta \phi(k)\delta \phi(-k)} e^{-\hat{S}_I - U} \right\rangle - \frac{1}{2} (2\pi)^4 \delta^{(4)}(0) \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{d\log G_\Lambda}{d\Lambda} \left\langle e^{-\hat{S}_I - U} \right\rangle.
\]

Here, the second term is a constant, and so we have
\[
\Lambda \frac{d}{d\Lambda} U = \frac{1}{2} V_4 \Lambda \frac{d}{d\Lambda} \int \frac{d^4k}{(2\pi)^4} \log G_\Lambda(k), \tag{21}
\]

where \( V_4 = (2\pi)^4 \delta^{(4)}(0) \), and so
\[
U(\Lambda) = U_0 + \frac{1}{2} V_4 \int \frac{d^4k}{(2\pi)^4} \log G_\Lambda(k) \tag{22}
\]

where \( U_0 \) is independent of \( \Lambda \), and
\[
\Lambda \frac{d}{d\Lambda} e^{-\hat{S}_I} = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \frac{\delta^2}{\delta \phi(k)\delta \phi(-k)} e^{-\hat{S}_I}, \tag{23}
\]

or, equivalently,
\[
\Lambda \frac{d}{d\Lambda} S_I = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Lambda \frac{dG_\Lambda(k)}{d\Lambda} \left[ \frac{\delta S_I}{\delta \phi(k)\delta \phi(-k)} - \frac{\delta^2 S_I}{\delta \phi(k)\delta \phi(-k)} \right]. \tag{24}
\]