Chapter 12

Problem Set Solutions

12.1 Problem Set 1 Solutions

1. \[ \vec{A} = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \] (12.1)

(a) \[ \nabla \times \vec{A} = \frac{1}{x^2 + y^2} \nabla \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \nabla \frac{1}{x^2 + y^2} \times \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = 0 \] (12.2)

except at \( x = y = 0 \) where \( \nabla \times \vec{A} \) is singular.

(b) For any closed path which does not wind around \( x = y = 0 \) line one gets

\[ \oint_C d\vec{S} \cdot \vec{A} = 0 \] (12.3)

because of above.

If \( C \) winds around the \( z - \alpha \times \beta \) one instead gets,

\[ \oint_C d\vec{S} \cdot \vec{A} = 2\pi \] (12.4)

We conclude that

\[ \nabla \times \vec{A} = 2\pi \delta(x)\delta(y)\hat{z} \] (12.5)

A way to realize the setup is a thin long solenoid at \( z - \alpha \times \beta \).
(c) 

\[ H = \frac{(\vec{p} + q\vec{A})^2}{2\mu} = \frac{p^2 + 2q\vec{A}\vec{p} + q^2\vec{A}^2}{2\mu} \]  

(12.6)

using \( \vec{\nabla} \times \vec{A} = 0 \).

We change to cylindrical cards and consider the wave-function

\[ \psi(\rho, z > 0) = \psi(\rho, z)e^{im\theta} \Rightarrow H = \frac{p_\rho^2}{2\mu} + \frac{p_z^2}{2\mu} + \frac{L_z^2}{2\mu\rho^2} + \frac{A_\theta L_z}{\mu} + \frac{q^2A^2}{2\mu} \]  

(12.7)

For \( \vec{A} = \frac{\hat{u}}{q} \) one gets

\[ H\psi = \frac{1}{2\mu}(p_\rho^2 + p_z^2 + \frac{1}{\rho^2}(\hbar + q)^2)\psi \]  

(12.8)

We see the contrufugel pet.

\[ V = \frac{1}{2\mu\rho^2}(\hbar + q)^2 \]  

(12.9)

unless \( q = \hbar m \) (in which case \( \psi \to \psi e^{im\theta} \)) we change the spectrum. For \( \psi \) to be single valued, \( \theta = \frac{2\pi \hbar}{q} \Rightarrow \text{flux quantization.} \)

(d) As \( \hbar \to 0 \) the dependence on \( \vec{A} \) of the spectrum gets away so this may quantum. As \( \hbar \to 0, V = \frac{1}{2\mu^2}q^2 \). A classical effect is that as you change the strength of \( A \), i.e., \( q \) spectrum changes continuously.

(e) If you Legendre transform you see that

\[ d \supset \frac{1}{2}\mu\rho^2\theta^2 - \theta\rho q A_\theta \]  

(12.10)

in cylindrical cards (there are other terms that \( L \) don’t better). A conserved charge associated with the solutions around the \( z = \alpha \times \beta \) is

\[ d_z = \frac{\partial d}{\partial \theta} = m\rho^2\theta - \rho q A_\theta \]  

(12.11)

This is canonical momentum. The mechanical momentum \( L_z + \rho q A_\theta \) is not necessarily conserved.

2. (a)

\[ A_{\mu\nu\rho} \to A_{\mu\nu\rho} + I_{[\mu\Lambda\nu\rho]} \]  

(12.12)
Define

$$F_{\mu\nu\rho\sigma} = I_{[\mu\lambda\nu\rho\sigma]}$$

(12.13)

Then

$$K.E. = -\frac{1}{2} \frac{1}{4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}$$

(12.14)

E.O.M.:

$$I_{\mu} F^{\mu\nu\rho\sigma} = 0$$

(12.15)

(Bronchi is trivial since there is no 5–index anti-symmetric tensor).

Easiest way to see the number of d.o.f. is to take Poincare dual:

$$F(x) = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}(x)$$

(12.16)

so there is one field degree of freedom off-shell.

On-shell one uses E.O.M.:

$$I_{\mu} F(x) = 0 \Rightarrow F(x) = F = \text{constant}$$

(12.17)

This is suggested to be associated with the cosmological constant: hep–th 0111032, hep–th 0005276

(b) $A_{\mu\nu\rho}$ couples to volume-form $dx^\mu \Lambda dx^\nu \Lambda dx^\rho$. The classical source coupling is $\int A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho$. Under a g.t. this changes as

$$\int_{I_{\mu}} A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho \rightarrow \int_{I_{\mu}} A_{\mu\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho + \int_{I_{\mu}} I_{\nu\rho} dx^\mu \Lambda dx^\nu \Lambda dx^\rho$$

(12.18)

We should require that

$$\int_{I_{\mu}} \Lambda_{\nu\rho} dx^\nu \Lambda dx^\rho = 0$$

(12.19)

where $I_{\mu}$ denotes the boundary of the shape it couples to. So either you require $\Lambda_{\nu\rho}(I_{\mu}) = 0$ to be the only sensible g.t.’s or you require $I_{\mu} = 0$ ($\mu$ is compact) (which solves the problem for arbitrary $\Lambda_{\nu\rho}$).
(c) E.O.M. gets modified as

\[ I_\mu F^{\mu\nu\rho\sigma} = J^{\nu\rho\sigma} \]  

(12.20)

Let’s find the source

\[ J^{\mu\nu\rho}(x) = \frac{\delta}{\delta A_\mu^\nu(x)} \int A_\alpha^\beta\gamma(y)dy^\alpha A dy^\beta \Lambda dy^\gamma \]  

(12.21)

To vary with respect to \( A_\mu(x) \) which lives on Minkewski, we should work out the embedding of \( \mu \) into Minkewski. Parameterize the space-time coordinate on the world-volume as \( y^\mu(u_1, u_2, u_3) \). Then above integral is

\[ \int_{\text{Minkewski}} d^4x A_\alpha^\beta\gamma(x) \det(\frac{dy^\alpha}{dx^\mu}) \delta(F(y)) \]  

(12.22)

where \( F(y) \) defines the surface

\[ J_{\alpha\beta\gamma}(x) = \int \delta^4(x - y(u_1, u_2, u_3))(\det \frac{dy^\alpha}{du^\nu}) d^3u \]  

(12.23)

(d) Important features are:

- \( B_\mu \) encodes all information in \( A_\mu^\nu. \)
- It has the gauge symmetry.
- \( B_\mu \rightarrow B_\mu = I_\nu A_\mu^\nu \)  

(12.24)

- \( F_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} \nabla \cdot B \)  

(12.25)

(e) Complete solution can be found in Peskin and Schroeder.

3. (a) To find a basis for \( SU(N) \) matrices parameterize the \( N \times N \) traceless and Hermitian matrix. In case of \( SU(3) \) this is

\[
\begin{pmatrix}
    a & b + ic & d + ie \\
    b - ic & f & g + ih \\
    d - ie & \rho - ih & -a - f 
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & -1 & 0 \\
    i & 0 & 1 
\end{pmatrix} +
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 1 & 0 
\end{pmatrix} +
\begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 
\end{pmatrix} +
\begin{pmatrix}
    0 & 0 & 1 \\
    0 & 0 & 0 \\
    0 & 0 & 0 
\end{pmatrix} +
\]
We read of the basis elements $\tilde{T}_a$ as coefficient of $a, b, \cdots, h$, requiring $tr w^a w^b = \frac{1}{2} \delta^{ab}$ means choosing $w^a = \frac{1}{2} \tilde{T}_a$. This is a nice basis.

(b) 

$$|\alpha|^2 + |\beta|^2 = 1 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2$$  \hspace{1cm} (12.27)$$

Therefore, topology of $SU(2)$ is $S^3$.

Topology of $SU(3)$ is an $S^3$ bundle over $S^5$ (see hep-th 9812006).

(c) For any representation of a Lie algebra $[T^a, T^b] = if^{abc}T^c$ one can get a conjugate representation by $\tilde{T}^a = -T^a$ because taking complex conjugate of the commutation relations give $[-T^{a*}, -T^{b*}] = if^{abc}(-T^{c*})$ for $f^{abc}$ real.

Since $T^a$ are Hermitian complex conjugate of a covariant vector transforms as contravariant vector.

A general tensor with $n$ upper, $m$ lower indices can be used to denote a general (might be reducible) representation: $\rho_{j_1 \cdots j_n}^{i_1 \cdots i_m}$ transfers as

$$\rho \rightarrow [T_a \rho]_{j_1 \cdots j_n}^{i_1 \cdots i_m} = \sum_{l=1}^n [T_a]_{i_l}^{j_l} \rho_{j_1 \cdots j_{l-1}j_{l+1} \cdots j_n}^{i_1 \cdots i_{l-1}i_{l+1} \cdots i_m} - \sum_{l=1}^m [T_a]_{i_l}^{i_l} \rho_{j_1 \cdots j_n}^{j_l \cdots j_{l-1}i_{l+1} \cdots i_m}$$  \hspace{1cm} (12.28)$$

From this transformation law, it is clear that one can impose symmetry among $(j_1 \cdots j_n)$ and $(i_1 \cdots i_m)$ and also one can impose tracelessness:

$$\delta_{j_1}^{i_1}\rho_{j_2 \cdots j_n}^{i_2 \cdots i_m} = 0$$

In fact every tensor with $n$ symmetric upper and $m$ symmetric lower index with the additional restriction of tracelessness corresponds to an irreducible representation.

$\delta_{i_l}^j$ transforms

$$[T^a \delta_{i_l}^j] = [T^a]_{j}^{k} \delta_{i_l}^{k} - [T^a]_{i_l}^{j} \delta_{k}^{k} = 0$$  \hspace{1cm} (12.29)$$

so it is invariant. $\epsilon_{i_1 i_2}$ transforms as

$$[T^a \epsilon_{i_1 i_2}] = [T^a]_{i_1}^{k} \epsilon_{ki_2} + [T^a]_{i_2}^{k} \epsilon_{ki_1}$$  \hspace{1cm} (12.30)$$

since $\epsilon$ is anti-symmetric only independent component is $\epsilon_{12}$
[\{T^a\}^\varepsilon]_{12} = [\{T^a\}]^1_{12} + [\{T^a\}]^2_{12} = \epsilon_{12} tr[\{T^a\}] = 0 \quad (12.31)

so \( \epsilon \) is invariant.

You can raise indices with \( \epsilon^{ij} \) so sufficient to consider only upper index tensors in \( SU(2) \). For a tensor \( \tau^{i_1i_2\cdots i_n} \) applying \( \epsilon_{ir_i} \) on the antisymmetric components give invariant subspaces. Hence totally symmetric requirements are irreducible. Dimension of \( \rho^{i_1\cdots i_n} \) (with \( i_1\cdots i_n \) symmetrized) can be found as follows: \( i_k \) runs over 1, 2. So linearly independent components of \( \rho \) are given by partitioning the set \( i_1\cdots i_n \) as \( 1111 \cdots 222 \cdots 2 \). The number of ways of doing this is the number of ways you can put one partition among \( n \) boxes, i.e., \( \binom{n+1}{1} = n+1 \) Note that this is the dimension of spin–\( \frac{n}{2} \) representation.

From the transformation law \( L \) gave above we see that

\[
[T_a\rho]^{j_1\cdots j_n} = \sum_{l=1}^n [T^a]^l_{jk} \rho^{j_1\cdots\tilde{j_l}j_{l+1}\cdots j_n} \quad (12.32)
\]

since \( T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and

\[
\rho^{i_1\cdots i_n} = \rho^{i_1} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \rho^{i_2} \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \rho^{i_n} \quad (12.33)
\]

where each covariant vector is a spin–\( \frac{1}{2} \) representation, \( T_3 \) reads the total \( S_z \) (\( z \) components of the spin) in the representation \( \rho^{i_1\cdots i_n} \). This is in the range \( \left( \frac{n}{2}, -\frac{n}{2} \right) \) so \( \rho^{i_1\cdots i_n} \) is indeed a spin–\( \frac{n}{2} \) representation and each state in this representation is labeled by the eigenvalue of \( T_3 \). Bells are ringing.

(d) Tensor products of representation of the group is \( R_1 \otimes R_2 \). Since group elements are obtained by erspenenrating the algebra \( G = e^T \), tensor products of the representation of the algebra are of the form \( r_1 \otimes 1_2 + 1_1 \otimes r_2 \). This obviously satisfy the same commutation relations.

Let me only show the evaluation of \( C_2(\rho) \) in the most non-trivial example, \( C_2(27) \) in \( SU(3) \). Consider the Clebsh-Gordon decomposition of a product representation:

\[
\rho_1 \otimes \rho_2 = \sum_i \rho_i \quad (12.34)
\]

The way \( T^a \) acts on \( \rho_1 \otimes \rho_2 \) is given above
12.1. PROBLEM SET 1 SOLUTIONS

\[ T^a_{\rho_1 \otimes \rho_2} = T^a_{\rho_1} \otimes 1_{\rho_2} + 1_{\rho_1} \otimes T^a_{\rho_1} \quad (12.35) \]

So

\[ tr(T^a_{\rho_1 \otimes \rho_2} T^a_{\rho_1 \otimes \rho_2}) = (C_2(\rho_1) + C_2(\rho_2))d\rho_1 d\rho_2 \quad (12.36) \]

On the other hand,

\[ T^a_{\rho_1 \otimes \rho_2} = \sum_i T^a_{\rho_i} \quad (12.37) \]

\[ tr(T^a_{\rho_1 \otimes \rho_2} T^a_{\rho_1 \otimes \rho_2}) = tr(\sum_i T^a_{\rho_i} \sum_j T^a_{\rho_j}) = \sum_i tr(T^a_{\rho_i} T^a_{\rho_j}) = \sum_i C_2(\rho_i)d\rho_i \quad (12.38) \]

Then,

\[ (C_2(\rho_1) + C_2(\rho_2))d\rho_1 d\rho_2 = \sum_i C_2(\rho_i)d\rho_i \quad (12.41) \]

27 occurs in the product of two 8’s:

\[ 8 \times 8 = 27 + 10 + 10 + 8 + 8 + 1 \quad (12.42) \]

You should have found that \( C_2(8) = 3, \ C_2(10) = 6 \). Plug these in:

\[ (3 + 3) \cdot 8 \cdot 8 = C_2(27) \cdot 27 + 2 \cdot 6 \cdot 10 + 2 \cdot 3 \cdot 8 + 0 \quad (12.43) \]
\[ 8 \cdot 8 \cdot 6 = 27C_2(27) + 168 \quad (12.44) \]
\[ C_2(27) = \frac{216}{27} = 8 \quad (12.45) \]