1. [4 Points] Convolution of Line Shapes

When two separate physical processes both contribute to the line shape, the resultant line shape is the convolution of the distributions. Say \( D_1 (\omega - \omega_0) \) and \( D_2 (\omega - \omega_0) \) are the normalized line shapes of the first and second type of process. The resultant line shape is then their convolution

\[
D_R (\omega - \omega_0) = \int_{-\infty}^{\infty} d\omega' \ D_1 (\omega - \omega') \ D_2 (\omega' - \omega_0)
\]

a) (1 point) Consider that \( D_1 \) and \( D_2 \) are both Lorentzian functions with FWHMs (full widths at half-maximum) \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Show that \( D_R \) is also Lorentzian and find its FWHM. (Hint: use the Fourier convolution theorem)

b) (1 point) Do the same if \( D_1 \) and \( D_2 \) are Gaussian and \( \Gamma_1 \) and \( \Gamma_2 \) are r.m.s. deviations.

c) (2 points) Find the FWHM of a Gaussian with r.m.s. deviation \( \Gamma/2 \). Find the r.m.s. deviation of a Lorentzian with FWHM \( \Gamma \).

2. [8 points] Intensity Distribution Due to Spontaneous Emission

An atom of total angular momentum \( J \) has a spontaneous radiation rate \( A \). It radiates to a lower level with angular momentum \( J' = J - 1 \). The problem is to find the rates for the various allowed transitions, i.e. the fraction of the radiation that goes into each of the possible transitions \((J, m) \rightarrow (J', m')\). The rates can be found by applying the following considerations:

- The sum of the rates out of each state \((J, m)\) must equal \( A \).
- The sum of the rates into each state \((J', m')\) must equal \( A \times \frac{2J+1}{2J'+1} \).
- An unpolarized mixture of radiators in level \( J \) must emit equal intensities of light with each of the three polarization components.
- The rate for a transition \((J, m) \rightarrow (J', m')\) must be the same as for \((J, -m) \rightarrow (J', -m')\).

For \( J = 2 \), \( J' = 1 \), designate the transitions by letters as follows:

- a: \( m = 2 \rightarrow m' = 1 \)
- b: \( m = 1 \rightarrow m' = 1 \)
- c: \( m = 0 \rightarrow m' = 1 \)
- d: \( m = 1 \rightarrow m' = 0 \)
- e: \( m = 0 \rightarrow m' = 0 \)

1. Find the rates for a through e, and present your results on a figure.
2. Find the rates for $a$ though $e$, using the Wigner-Eckart theorem. (Clebsch-Gordan coefficients can either be worked out from first principles, or taken from a table in one of the quantum mechanics or spectroscopy texts.)

Note that the transition rates calculated here are important in experiments involving laser excitation. Because emission and absorption rates are proportional, the distribution of emission rates yields the relative strengths of the transitions, i.e. their relative rates of excitation.

3. [8 points] Saturation Spectroscopy
This problem guides you through the concepts of saturation spectroscopy. This is one of the techniques to perform Doppler-free spectroscopy, i.e. to extract a narrow line (with the natural linewidth) in a gas with a broad velocity distribution. It nicely illustrates the combination of homogenous and inhomogeneous line broadening. Saturation spectroscopy is frequently used to lock lasers to atomic lines. You should not get into nasty integrals for this problem. The drawn lineshapes should clearly show the basic features, but don’t have to be exact.

a) Homogeneous broadening
Consider a dilute gas of density $n$ composed of atoms with resonant frequency $\omega_0$ and linewidth $\Gamma$. The gas is exposed to monochromatic light of frequency $\omega_L = \omega_0 + \delta$ and intensity $I = sI_{\text{SAT}}$ where $I_{\text{SAT}}$ is the saturation intensity. Let us ignore the effects of the motion of the atoms, i.e. consider temperature $T = 0$. What are the densities of atoms in the ground state $n_1$ and the excited state $n_2$, including the effect of saturation? What is the cross-section for absorption? The gas is in a box of length $L$ along the direction of the incoming light. What fraction of the light is absorbed? (This is a small fraction, so don’t worry about the effect of light attenuation on the saturation of the sample).

b) The Bennet hole
Now, let’s endow these atoms with a mass $m$ and a temperature $T$. Let the incoming light have a wavevector $k_L$ along the z-axis. What is the population density distribution in the ground state $n_1(v_z)$ as a function of $v_z$, the component of velocity in the z-direction? You should find that the light “burns a hole” (known as the Bennet hole) in the distribution of absorbers. What is the position of the hole? How do the width and depth (relative to the population for $s = 0$) vary with saturation parameter $s$?

c) Inhomogeneous broadening
Consider that we sweep the frequency of the incident laser $\omega_L$ and measure the (small) absorption of the beam. Determine the fraction of the light absorbed as a function of $s$ and $\delta$ and compare with its value at $s = 0$ (you don’t need to solve the integral). For high temperatures ($k_L \bar{v}_z \gg \Gamma$), what is the width of the absorption line? Does saturation affect the width?

d) Saturation spectroscopy
To actually get some benefit from saturating the gas, we introduce a second laser beam. i) We can add a weak probe beam at frequency $\omega_p$ with wavevector $k_p$. What is the absorption of this beam, including the effects of the saturating beam ($k_L, \omega_L$)? Again, just write the integral, and take the length of the box along $k_p$ to be $L$. Draw the absorption line shape, identifying the position and width of its features. ii) We can also just retroreflect the original beam. Draw the population distribution $n_1(v_z)$ for $\omega_L \neq \omega_0$ and $\omega_L = \omega_0$. Identifying the depth of the Bennet hole(s), draw the lineshape of absorption of the retroreflected beam (i.e. we scan $\omega_L$). What is the width of the central feature (at $\delta = 0$)?