1. Consider a two dimensional superconductor with a $d$-wave energy gap given by

$$\Delta(\phi) = \Delta_0 \cos 2\phi .$$

Assume an isotropic energy band with Fermi velocity $\nu_F$ in the normal state. The quasiparticle spectrum is given by

$$E(k) = \sqrt{\nu_F^2 (|k| - k_F)^2 + \Delta^2(\phi)} .$$

(a) Show that the energy gap vanishes at 4 points on the Fermi surface. In the vicinity of these nodal points, show that the quasiparticle dispersion is given by

$$E(k) = \sqrt{\nu_F^2 k_1^2 + \nu_2^2 k_2^2} ,$$

where $k_1$ and $k_2$ are momentum components perpendicular and parallel to the Fermi surface measured from the nodal points. What is $\nu_2$ in terms of $\Delta_0$ and $k_F$ ? Show that the density of states at energy $E$ per node per spin is $\frac{1}{2\pi\nu_F\nu_2} E$.

(b) Show that at low $T$, thermal excitation of the quasiparticles leads to a linear $T$ reduction of the superfluid density

$$\frac{\rho_s(T)}{m} = \frac{\rho_s(T = 0)}{m} - \frac{2 \ln 2 \nu_F}{\pi} T .$$

The integral you encounter can be done by a change of variable $y = e^{-x}$.

(c) In the presence of $A$ and $\nabla \theta$ where $\theta$ is the phase of the order parameter, the quasiparticle spectrum is changed by

$$E(k, A) = E(k) + \nu_F \cdot \frac{1}{2} \left( \nabla \theta + \frac{2e}{c} A \right)$$

The last term is the gauge invariant generalization of the term we discussed in class. Consider a single vortex and assume the superconductor is extreme type II.
At a distance $R$ away from the vortex core in the $\hat{x}$ direction, calculate the density of states which is generated at the Fermi level. (Assume $\xi_0 \ll R \ll \lambda_L$.) How is your answer different if you approach the vortex core in the $(1,1)$ direction?

(d) In an external field $H$, a triangular vortex lattice is formed. Show that the density of states found in (c) gives rise to the following unusual contribution to the specific heat

$$c_\nu = \alpha \sqrt{HT},$$

Make a crude estimate of the coefficient $\alpha$.


2. Make a table for the real part of the transverse and longitudinal response functions $K_\perp$ and $K_\parallel$. Give the limits $\omega = 0, q \to 0$, and $q = 0, \omega \to 0$ for a perfect metal, a disordered metal, and a superconductor with or without disorder (16 quantities in all!). Write the leading nonvanishing contributions in terms of physical quantities such as Landau diamagnetism, conductivity and scattering lifetime.