LECTURE V

Continued discussion on Kubo formula:

**Sanity Check** with a random potential:

\[
\bar{V}(\mathbf{r})V(\mathbf{r'}) = V_0^2 \delta(\mathbf{r} - \mathbf{r'})
\]

\[
\sigma_{\mu\nu}(q, \omega) = \frac{1}{\omega} \int d(\mathbf{r} - \mathbf{r'}) \int dt e^{-i\omega t} \langle [j_{\mu\nu}(\mathbf{r}, t), j_{\nu\mu}(\mathbf{r'}, t)] \rangle |0 >
\]

\[
= \frac{1}{\omega} \frac{1}{\Omega} \int d\mathbf{r} \int d\mathbf{r'} \int dt e^{-i\omega t} \langle [j_{\mu\nu}(\mathbf{r}, t), j_{\nu\mu}(\mathbf{r'}, t)] \rangle |0 >
\]

Where the bar in the above equation is the *impurity average*.

**DC** \( q = 0 \), \( \omega \rightarrow 0 \)

\[
\sigma(\omega) = \frac{1}{\omega} \frac{1}{\Omega} \sum_n <0 | \int d\mathbf{r} j_{\mu\nu}(\mathbf{r}) | n > < n | \int d\mathbf{r'} j_{\nu\mu}(\mathbf{r'}) | 0 > \delta(\omega - (E_n - E_0))
\]

Where \( |n > \) is an exact eigenvalue of the full *one-body* hamiltonian: \( (H + V)|n > = E_n |n > \) In principle one can find the spectrum of \( (H + V) \) so \( |n > \) is the particlehole pair:

\[
|n > = |\beta\alpha >, E_\beta > E_F, E_\alpha < E_F
\]

\[
\int d\mathbf{r} < n | j_{\mu\nu}(\mathbf{r}) | 0 > = \int d\mathbf{r} \left( \frac{e}{m} \right) < n | \nabla_\mathbf{r} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) | 0 >
\]

\[
= \int d\mathbf{r} \left( \frac{e}{m} \right) \int d\mathbf{r}_1 \varphi_\beta^*(\mathbf{r}_1) \nabla_\mathbf{r} \delta(\mathbf{r} - \mathbf{r}_1) \varphi_\alpha(\mathbf{r}_1)
\]

\[
= \frac{e}{m} \int d\mathbf{r}_1 \varphi_\beta^*(\mathbf{r}_1) \nabla_\mathbf{r}_1 \varphi_\alpha(\mathbf{r}_1) = (e/m) < \beta | \nabla | \alpha >
\]
\[
\sigma(\omega) = \frac{\pi e^2}{\omega \Omega} \sum_{\alpha \beta} |< \beta | \frac{\nabla}{m} | \alpha >|^2 \delta(\omega - (E_\beta - E_\alpha)) f(E_\alpha)(1 - f(E_\beta))
\]

Where the "broken" average comes from our assumption that the random potential causes uncorrelated \(\varphi\) and \(E\).

\[
\sigma(\omega) = \frac{\pi e^2}{m^2 \omega \Omega} \int_{-\omega}^{\omega} d\omega_1 \sum_\alpha \delta(\omega_1 + E_\alpha) \sum_\beta \delta(\omega_1 - E_\beta - \omega_1) |< \beta | \frac{\nabla}{m} | \alpha >|^2
\]

MORE APPROXIMATIONS:

\[
|< \beta | \frac{\nabla}{m} | \alpha >|^2 = \nu^2
\]

\(\omega \gg\) level spacing \(\Rightarrow\) \(N(\omega) = \sum_\alpha \delta(\omega_1 + E_\alpha) \approx N(0)\)

Note that \(N(0)\) is the density of state per (just!) unit energy, so it should diverge as \(\Omega \to \infty\) or \(\Delta \to 0 : N(0) \approx \frac{1}{\Delta}\).

\[
\sigma(\omega \to 0) = \frac{e^2 \pi N_0^2}{\Omega \nu^2}
\]

Next we estimate \(|< \beta | \frac{\nabla}{m} | \alpha >|^2 :\)

Define \(l\) as the distance that wavefunction loses information about its phase so for a perfect plane wave (without scattering) \(l \to \infty\) and for a very strong scattering impurity \(l \to k_F^{-1}\). At this point we intend to find approximation for the \(|< \beta | \frac{\nabla}{m} | \alpha >|^2\) in the former regime or

\(k_F l \ll 1\)

Let’s make a grid out of our sample where each section has the volume of

\[
\nu = \frac{4\pi}{3} l^3
\]
Definition of l suggests that within each box we can in principle associate a wave vector to our wavefunction: 
So one an define
\[ \delta_i \equiv \int d\mathbf{r} \psi_k^* \frac{1}{m} \frac{\partial}{\partial x} \psi_k \]
associate k with \( \alpha \) and \( k' \) with \( \beta \). By this partitioning we have:
\[ \overline{v^2} = \overline{v_{\alpha\beta}^2} = \frac{\Omega}{\nu} \delta_i^2 \]
\[ \delta_i \approx \left( \int_0^\nu d\mathbf{r} \frac{e^{i(k-k')r} k}{\Omega m} \right) e^{i\phi_i} \]
Where \( \phi_i \) is the random phase at the site \( i \).

\[ |k - k'| = 2k_F \sin \frac{\theta}{2} \simeq k_F \theta \]
for \( k_F \theta l > 1 \) we will encounter rapid oscillations and \( \delta_i = 0 \) and for
\[ k_F \theta l < 1 \Rightarrow \delta_i = \frac{k_F \nu}{m \Omega} e^{i\phi_i} \]
In order to average over different boxes, we average over k and k’ which amounts to average over \( \theta \).

Now the \( \theta \) integration:
\[ \overline{v^2} = \frac{\Omega}{\nu} \left( \frac{k_F \nu}{m \Omega} \right)^2 \int_0^{1/kl} d\theta \frac{2\pi \sin \theta}{\pi} \]
\[ \int_0^{1/kl} d\theta \frac{2\pi \sin \theta}{\pi} = \frac{1}{4(kl)^2} \]
\[ \therefore \overline{v^2} = \frac{\pi l}{3m^2 \Omega} \]
\[ \sigma(\omega \to 0) = \frac{e^2 \pi^2}{3m^2} \left( \frac{N(0)}{\Omega} \right) l \]
Put \( l = \tau v_f \); \( n = \frac{k_F^3}{8\pi^2} \), \( \frac{N(0)}{\Omega} = \frac{m k_F}{2\hbar^2 \pi^2} \) in the
\[ \sigma_{\text{Boltzmann}} = \frac{ne^2 \tau}{m} \]
you’ll get the same result (Of course with different coefficient) \sim
\frac{e^2}{\hbar} k_p l