Path integrals for many particle systems

If the particles are all distinguishable, then extension to many particles is completely trivial - simply integrate over the trajectories of the coordinates $(x_1, \ldots, x_N)$.

What if the particles are identical?

To bring out the issues associated with "statistics" of identical quantum particles, consider the case of just 2 particles with coordinates $(\mathbf{x}_1, \mathbf{x}_2)$ ($\mathbf{x}_1, \mathbf{x}_2$ are vectors in d-dimensions).

Consider the amplitude for an initial configuration to return to the same configuration after a time $t$.

To also clearly separate issues of statistics from more ordinary short ranged interactions, assume that "giant-core" there is a strong repulsion when the particles are very close to each other so that they always stay a non-zero
distance apart.

Then we can discuss the motion in terms of center-of-
mass \( \mathbf{R} \) and relative coordinates \( \mathbf{r} \).

Again issues of statistics & the effect of exchange only
enter the relative coordinate \( \mathbf{r} \) - so we will focus
on this entirely & ignore the CM coordinate \( \mathbf{R} \).

Note that \( \mathbf{r} = 0 \) is excluded by the hard-core
restriction so \( \mathbf{r} \in \mathbb{R}^d - \{0\} \).

Paths where the initial & final 2-particle configurations
are the same have either \( \mathbf{r}_{\text{fin}} = \mathbf{r}_{\text{ini}} \)
or \( \mathbf{r}_{\text{fin}} = -\mathbf{r}_{\text{ini}} \).

Generally the amplitude for any particular path \( \mathbf{P} \)

\[ A[\mathbf{P}] \propto e^{iS[\mathbf{P}]} \]

where \( S[\mathbf{P}] \) is the classical action evaluated
along the path.
For all paths that can be continuously deformed into one another, the constant of proportionality is the same.

However for 2 paths that cannot be continuously deformed into one another, the amplitude can (in principle) differ by an overall phase.

Write \( A[p] = (e^{i\theta})(e^{iS[p]}) \)

where \( \theta \) is fixed for within each class of paths that can be continuously deformed into one another.

For \( U=3 \) (or higher), further if a path \( P_3 = P_2 P_1 \)

i.e. it is composed out of the 2 individual return paths \( P_2 \) \& \( P_1 \), then

\[ e^{i\theta P_3} = e^{i\theta P_1} e^{i\theta P_2} \]
For $d=3$ (or higher), there are only 2 classes of paths that cannot be continuously deformed into one another:

- Paths for which $\vec{r}_f = \vec{r}_i$ form one class.
- Paths with $\vec{r}_f = -\vec{r}_i$ form another class.

As the overall phase choice is arbitrary, choose $\Theta = 0$ for paths where $\vec{r}_f = \vec{r}_i$, and let

$\Theta = \Theta$ for paths where $\vec{r}_f = -\vec{r}_i$.

Consider paths which go from north pole to south pole and then back again to north pole, we get

$e^{2i\Theta} = 1 \Rightarrow e^{i\Theta} = \pm 1$.

$e^{i\Theta} = +1$ describes bosons,

$e^{i\Theta} = -1$ describes fermions.

In $d=2$ situation is different.
Characterize $\mathbb{C}$ by polar coordinates

$$(r, \phi)$$

The important variable is $\phi$.

Return paths have $\phi = 0$ or $\pi$ (mod $2\pi$).

However, every path where $\phi$ starts from $0$ and winds around several times to reach $m \pi$, is distinct (for distinct $m$).

There are an infinite number of classes of paths that cannot all be deformed into one another (each class is labelled by the winding $m$).

If for $m = 0$ we choose $\phi = 0$, and for $m = 1$, set $\phi = \theta$, then $\phi_m = m \phi$.

Thus we can have a perfectly consistent quantum theory where exchange of 2 particles ($m = 1$) gives a phase factor which is neither 0 nor $2\pi$ in $d = 2$. 
Such particles are known as "anyons".

Returning to bosons/fermions, and in the case of arbitrary $N$ of total $N$ particles,

$$Z = \text{Tr}(e^{-\beta H})$$

$$\zeta_{\text{partition}} = \sum \int_{\mathbb{R}^N} dx_1 \cdots dx_N \left| \sum_{i=1}^N \langle x_i | e^{-\beta H} | \{x_i, \cdots, x_N\} \rangle \right|^2$$

$$= \frac{1}{N!} \int_{\mathbb{R}^N} dx_1 \cdots dx_N \sum_{\pi} \text{sgn}(\pi) \left| \sum_{i=1}^N \langle x_{\pi_i} | e^{-\beta H} | x_{\pi_{i+1}} \rangle \right|^2$$

where $\pi$ now denotes a permutation.

Can now proceed as before (break up $e^{-\beta H}$ into $(e^{-\beta H})^M$),

insert resolution of identity

$$1 = \int dx_1 \cdots dx_N | x \cdots x \rangle$$

$$\langle x \cdots x |$$

$M$ to get.
\[ Z = \frac{1}{N!} \sum_p \Sigma p_{m}(p) \int \left[ \mathcal{D}r_1 \mathcal{D}r_2 \cdots \mathcal{D}r_N \right] \]
\[ x_1(p) = x_1(0) \]
\[ x_N(p) = x_N(0) \]
\[ e^{\int_{\beta} \mathcal{L}} \]
\[ \mathcal{L} = \sum_{i=1}^{N} \left( \frac{m}{2} \left( \frac{dx_i}{dt} \right)^2 + U(x_i) \right) \]

(assuming the Hamiltonian is)
\[ H = \sum_i \frac{p_i^2}{2m} + U(x_i) \]

Coherent state path integral

Focus on bosons (deal with fermions at a later stage when necessary).

First review coherent states - these provide a useful basis for states in Fock space.

They are eigenstates of annihilation operators.

Consider a single particle state \([a] \) and the corresponding destruction operator \(a\)
Define \( |\phi\rangle \) such that \[ a_x |\phi\rangle = \phi_x |\phi\rangle \]

First consider states where exactly 1 level \( x \) is occupied. Then let \[ |\phi\rangle = \sum_n \phi_n |n_x\rangle \]

\[ a_x |\phi\rangle = \sum_n \phi_n |n_{x-1}\rangle = \phi_x |\phi\rangle \]

\[ \Rightarrow \phi_x \phi_n = \sqrt{n_x} \phi_n \]

\[ \Rightarrow \phi_{n_x} = \phi_x \sqrt{n_x} \]

so that \[ |\phi\rangle = \sum_n \frac{\phi_x \sqrt{n_x}}{\sqrt{n_x}} |n_x\rangle \]

In general expand \[ |\phi\rangle = \sum_n \phi_n |n_x\rangle \]

Following same line of reasoning as above, conclude
\[ \{n_{\alpha}\} = \prod_a \frac{\phi_n^a}{\sqrt{n_a!}} \text{ so that} \]

\[ |\psi\rangle = \sum_{\{n_{\alpha}\}} \left( \frac{\phi_1^{n_1}}{\sqrt{n_1!}} \frac{\phi_2^{n_2}}{\sqrt{n_2!}} \ldots \right) |\{n_{\alpha}\}\rangle \]

Now use \( |\{n_{\alpha}\}\rangle = \left( \frac{\phi_1^{n_1}}{\sqrt{n_1!}} \frac{\phi_2^{n_2}}{\sqrt{n_2!}} \ldots \right) |0\rangle \) to write

\[ |\psi\rangle = \sum_{\{n_{\alpha}\}} \left( \phi_1 a_1^{n_1} \phi_2 a_2^{n_2} \ldots \right) |0\rangle \]

\[ = e\left( \phi_1 a_1 + \phi_2 a_2 + \ldots \right) |0\rangle \]

\[ |\psi\rangle = e^{\sum \phi_n^a a^+_a} |0\rangle \]

\[ \langle \psi | = \langle 0 | e^{\sum \phi_n^a a^+_a} \]

\[ \langle \psi | = \sum \phi_n^a a^+_a \]
The overlap \( \langle \phi | \phi' \rangle \) of two coherent states is

\[
\langle \phi | \phi' \rangle = \sum_{\{n_1', n_2'\}} \sum_{\{n_1, n_2\}} \frac{(\phi^*_{n_1', \alpha_1'} \phi_{n_2', \alpha_2'})}{\sqrt{n_1'n_2'}} \frac{(\phi^*_{n_1, \alpha_1} \phi_{n_2, \alpha_2})}{\sqrt{n_1n_2}}
\]

\[
= \sum_{\{n_1', n_2'\}} \sum_{\{n_1, n_2\}} \frac{1}{\sqrt{n_1'n_2'}} \frac{1}{\sqrt{n_1n_2}} \langle \sum_{\{n_1, n_2\}} | \sum_{\{n_1', n_2'\}} \rangle
\]

\[
= e^{\alpha^* \phi^* \alpha \phi'}
\]

The coherent states are an (over) complete basis.

\[
\int d\phi^* d\phi e^{-\frac{1}{2} \alpha^* \phi^* \alpha \phi} |\phi\rangle \langle \phi'| = 1
\]
Proof:

\[ \text{LHS} = \int \prod_x (\frac{d\phi_x^* d\phi_x}{2\pi i}) e^{-\frac{\xi}{\alpha} |\phi_x|^2} \]

\[ = \sum_{n,n'} \prod_x \frac{\phi_x^{n_x} (\phi_x^{\dagger})^{n'_x}}{\sqrt{n_n!}} \langle \phi_n | \bar{\phi}_{n'} \rangle \]

\[ = \sum_{n,n'} \int \prod_x \left[ \frac{d\phi_x^* d\phi_x}{2\pi i} e^{-\frac{\xi}{\alpha} |\phi_x|^2} \right] \frac{\phi_x^{n_x} (\phi_x^{\dagger})^{n'_x}}{\sqrt{n_n!}} \langle \phi_n | \bar{\phi}_{n'} \rangle \]

For any operators \( \hat{O} \),

\[ \text{tr} \hat{O} = \text{tr} \left[ \int d\phi^* d\phi e^{-\frac{\xi}{\alpha} |\phi|^2} \hat{O} \right] \]

\[ = \sum_{n} \langle \phi_n | d\phi^* d\phi e^{-\frac{\xi}{\alpha} |\phi|^2} | \phi_n \rangle \]

\[ = \langle \phi | \hat{O} | \phi \rangle \]
Consider any "normal-ordered" operator

\[ \hat{O} = \alpha^\dagger \alpha \beta \]

\[ \langle \phi | \alpha^\dagger \alpha \beta | \phi \rangle = \phi^\dagger \alpha \phi \beta \langle \phi | \phi \rangle = \phi^\dagger \phi \beta e^{i \phi^2} \]

In general if \[ \hat{O} = \left( \prod_\alpha (\alpha^\dagger \alpha)^{M_\alpha} \right) \left( \prod_\beta (\beta^\dagger \beta)^{M_\beta} \right) \]

\[ \langle \phi | \hat{O} | \phi \rangle = \left( \prod_\alpha (\phi^\dagger \alpha)^{M_\alpha} \right) \left( \prod_\beta (\phi \beta)^{M_\beta} \right) \langle \phi | \phi \rangle \]

Partition function \[ Z = \text{tr} \left( e^{-\beta \hat{H}} \right) \]

\[ = \int \prod_\alpha d\phi^\dagger \alpha d\phi \alpha \beta e^{-\frac{1}{2} \sum_\alpha \phi^2 \alpha} \langle \phi | e^{-\beta \hat{H}} | \phi \rangle \]

As before, develop a path integral representation for \[ Z \]
Proceed as usual

\[ Z = \int \frac{d\phi^* d\phi}{2\pi i} e^{-\phi^* \phi} \langle \phi | e^{-\epsilon H} | \phi \rangle \]

\[ N \epsilon = \beta, \quad \epsilon \to 0, \quad N \to \infty \]

\[ = \int \frac{d\phi^*_0 d\phi_0}{2\pi i} \prod_{j=1}^{N} d\phi^*_j d\phi_j \epsilon_0 \epsilon_j \exp \left(-\sum_{j=0}^{N} \phi^*_j \phi_j \right) \]

\[ \langle \phi_0 | e^{-\epsilon H} | \phi_N \rangle \langle \phi_N | e^{-\epsilon H} | \phi_{N-1} \rangle \]

\[ \langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle \ldots \]

\[ \langle \phi_i | e^{-\epsilon H} | \phi_0 \rangle. \]

\[ \langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle \approx \langle \phi_{j+1} | (1 - \epsilon H) | \phi_j \rangle + o(\epsilon^2) \]

Assume \( H = H(\{a^+_k\}, \{a_k\}) \) is normal ordered.

Then \( \langle \phi_{j+1} | H | \phi_j \rangle = \langle \phi_{j+1} | \phi_j \rangle \]

\[ = Ce^{-\phi} = e^{\phi} \langle \phi_{j+1} | \phi_j \rangle H(\phi^*_j, \phi_j) \]
\[ \langle \phi_{j+1} | e^{-\epsilon H} | \phi_j \rangle = e^{\epsilon \sum_{j=0}^N (\phi_{j+1} - \phi_j)} - \epsilon \sum_{j=0}^N H(\phi_{j+1}, \phi_j) + o(\epsilon). \]

\[ Z = \int \prod_{j=0}^N d\phi_j e^{-\epsilon \sum_{j=0}^N (\phi_{j+1} - \phi_j)} - \epsilon \sum_{j=0}^N H(\phi_{j+1}, \phi_j) \quad \left[ \phi_{NH} = \phi_0 \right] \]

Now introduce "continuum" notation when \( \epsilon \rightarrow 0, N \rightarrow \infty \).

Write \( \phi_{j+1} - \phi_j = \epsilon \frac{d\phi}{dx} \) to write

\[ Z = \int \mathcal{D}\phi \ e^{-\epsilon \int \frac{d^3 \phi}{dx} \left( \frac{d\phi}{dx} + H(\phi^*) \phi \right)} \quad \phi(\xi) = \phi(0) \]

= functional integral over \( \phi \) with periodic boundary conditions.
Comments: To be really precise, the meaning of continuum path integral is really a "short-hand" notation for the full discrete time integrals.

Strictly speaking, in discrete time,

\[ \sum_{j=0}^{N} \phi_{j+1} (\phi_{j+1} - \phi_j) \]

may be written in terms of the Fourier components of \( \phi \):

\[ \phi_j = \frac{1}{\beta} \sum_{n} e^{-i \omega_n^T} \phi(\omega_n) \]

with \( \omega_n = \frac{2 \pi n}{\beta} \), and \( T = j \epsilon \).

Then

\[ \sum_{j=0}^{N} \phi_{j+1} (\phi_{j+1} - \phi_j) = \frac{1}{\beta} \sum_{n} \phi(\omega_n)(1 - e^{-i \omega_n^T}) \phi(\omega_n) \]

In the continuum expression we have effectively replaced \( 1 - e^{-i \omega_n^T} \) by \( i \omega_n^T \).

Clearly this is legitimate only if \( \omega_n^T \leq 1 \).
ie if the "frequencies" of interest \( \omega \), are

\[ \text{temporal lattice spacing} \]

As in evaluation of \( Z \) for simple harmonic oscillators,
expect that for full calculation of \( Z \), all \( \omega \) contribute.

\( \omega \) we must be careful about sticking to the precise
lattice definition of the path integral.

However if for a general many body \( H = H_0 + H_{\text{int}} \)

where \( H_0 \) is the "free" quadratic part

2 Hint is the interacting part, we calculate

\[
\frac{Z}{Z_0} = \frac{\text{tr} \left( e^{-\beta H} \right)}{\text{tr} \left( e^{-\beta H_0} \right)} \quad \text{then expect that}
\]

This will only be dominated by \( \omega \), \( \omega \) so

the calculation can be formulated directly in the
continuum without worrying about carefully taking the
limit of the lattice action.
Similarly in calculating averages

$$\langle \hat{O} \rangle = \frac{\text{Tr}(\hat{O} e^{-\beta \hat{H}})}{\text{Tr}(e^{-\beta \hat{H}})}$$

expect can typically
directly work in the continuum.

**Real time path integral**

It is equally easy to derive a coherent state
path integral for the evolution operator $e^{-iHt}$ in
real time.

The result is

$$\langle \phi_f^* t_f | U | \phi_i^* t_i \rangle$$

$$= \int D\phi^* D\phi \ e^{i\int_{t_i}^{t_f} dt \left[ \frac{i\hbar}{\tau} \phi^* \partial \phi - H(\phi^*, \phi) \right]}$$

$$\phi(t_f) = \phi_f$$

$$\phi(t_i) = \phi_i$$
Phenomenon of superfluidity.

At ordinary (atmospheric) pressure, He-4 remains a liquid down to $T = 0$ K.

This is due to a combination of 2 things:

- (a) the He-4 atom is inert as it has a filled shell $\Rightarrow$ the residual interaction between two He-4 atoms is very weak.

- (b) He-4 is a very light atom $\Rightarrow$ kinetic energy at zero point motion is large.

However at low $T \approx 2$ K, He-4 nevertheless undergoes a phase transition from the higher temp. ordinary liquid to a lower temp. extraordinary liquid called the superfluid. 

\[
\text{[Phase dia: } \quad \begin{array}{c}
\text{Solid} \\
\text{Liquid} \\
\text{Gas}
\end{array} \quad \]

In the low-T phase, He-4 can flow through fine capillaries without any friction (i.e., no viscosity).

There are a number of other weird phenomena which can be summarized by the following description:
In the low-T phase, the fluid is a mixture of two components: a "normal" fluid and a "superfluid". The superfluid component has zero viscosity and zero entropy.

As an example of a phenomenon which suggests this, consider two containers of He-4 connected together by a fine capillary.

If $P_A > P_B$, the superfluid component will flow from A to B. As this superfluid component carries no entropy, the entropy/mass of A will increase and that of B will decrease.

$\Rightarrow$ A will heat up and B will cool down which is actually observed.

The inverse effect is also observed.

Consider heating A relative to B - the temp. gradient causes a pressure gradient forcing the superfluid component...
To observe the flow from A to B.

If the set up is like

\[ \text{heat} \rightarrow \text{packed with powder} \]

then the increased pressure in the heated region can cause superflow up the tube to create a fountain.

In searching for a theoretical understanding and description of superfluidity, we first note 2 things:

(a) The phenomenon happens at low-T where we expect that the quantum statistics of the He-4 atoms must play an important role.

(b) He-4 atoms are bosons— if we completely ignore the repulsion between He-4 atoms, we get the BEC phenomenon where there is indeed the formation of a zero entropy condensate.

(Note however BEC ≠ superfluidity).
Here we take a "top-down" approach and
attempt to describe the properties of a system
of interacting bosons at low temperature.

We will analyse a particular physical situation
which will enable a "painless" description of the
low-T superfluid phase using ideas of broken
symmetry.

The universal properties of the resulting phase
generalize to other physical situations.