Problem 1:  The T2K Experiment[50 points]

The T2K experiment uses an off-axis $\nu_\mu$ beam from $\pi^+ \rightarrow \mu^+ \nu_\mu$ decays. Consider the case where the pion has velocity $\beta$ along the $z$-direction in the laboratory frame and a neutrino with energy $E^*$ is produced at an angle $\theta^*$ with respect to the $z'$-axis in the $\pi^+$ rest frame.

(a) Show that the neutrino energy in the pion rest frame is $p^* = (m_\pi^2 - m_\mu^2) / 2m_\pi$.

(b) Show that the energy $E$ and angle of the production $\theta$ of the neutrino in the laboratory frame are $E = \gamma E^*(1 + \beta \cos \theta^*)$ and $E \cos \theta = \gamma E^*(\cos \theta^* + \beta)$ where $\gamma = E_{\pi} / m_\pi$.

(c) Using the expressions for $E^*$ and $\theta^*$ in terms of $E$ and $\theta$, show that $\gamma^2 (1 - \beta \cos \theta)(1 + \beta \cos \theta^*) = 1$.

(d) Show that the maximum value of $\theta$ in the laboratory frame is $\theta_{max} = 1 / \gamma$.

(e) In the limit $\theta \ll 1$ show that $E \approx 0.43 E_{\pi} \frac{1}{\sqrt{1 + \beta^2 \gamma^2 \theta^2}}$ and therefore on-axis ($\theta = 0$) the neutrino energy spectrum follows that of the pions.

(f) Assuming that the pions have a flat spectrum in the range 1-5 GeV, sketch the form of the resulting neutrino energy spectrum at the T2K far detector (Super-Kamiokande), which is off-axis at $\theta = 2.5^\circ$. Given that the Super-Kamiokande detector is 295 km from the beam, explain why this angle was chosen.
Problem 2: Nuclear Stability [30 points]

The Weizäcker formula or semi-empirical mass formula is a parametrization of nuclear mass as a function of $A$ and $Z$. Following this formula, the mass of an atom with $Z$ protons and $N$ neutrons is given by the following:

$$M(A, Z) = NM_n + ZM_p + Zm_e - a_V A + a_s A^{2/3} + a_c \frac{Z^2}{A^{1/4}} + a_a \frac{(N-Z)^2}{4A} + \frac{\delta}{A^{1/2}}$$

with $N = A - Z$.

The exact values of the parameters $a_V, a_s, a_c, a_a$, and $\delta$ depend on the range of masses for which they are optimized. One possible set of parameters is given by the following:

- $a_V = 15.67$ MeV/$c^2$
- $a_s = 17.23$ MeV/$c^2$
- $a_c = 0.714$ MeV/$c^2$
- $a_a = 93.15$ MeV/$c^2$
- $\delta = -11.2, 0, +11.2$ MeV/$c^2$ for even $Z$ and $Z$, odd $A$, or odd $Z$ and $N$, respectively.

For fixed $A$ find the proton number $Z$ for the most stable nucleus, and plot $Z$ as a function of $A$. Each term captures an aspect of the atom. Explain briefly how the individual terms can be interpreted.

Problem 3: Decay time dating [20 points]

Naturally occurring uranium is a mixture of the $^{238}$U (99.28%) and $^{235}$U (0.72%) isotopes.

How old must the material of the solar system be if one assumes that at its creation both isotopes were present in equal quantities? The lifetimes are $\tau(^{235}$U) = $1 \times 10^9$ years and $\tau(^{238}$U) = $6.6 \times 10^9$ years.

How much of the $^{238}$U has decayed since the formation of the earth’s crust $2.5 \times 10^9$ years ago?