Portfolio Risk Management

Summer 2003
Hot Off the Press

Please see the article:

Question: Given the choice to invest $1 million in these three firms, which one(s) would you choose?

<table>
<thead>
<tr>
<th>Asset</th>
<th>Expected Return (% per year)</th>
<th>Risk (% per year)</th>
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<tbody>
<tr>
<td>Exxon</td>
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</table>
Outline

- Covariance and correlation
- Sums of random variables
- Asset diversification in investing
Covariance and Correlation

How do we describe the relationship between two random variables?

Example: Chain of upscale cafés sells gourmet hot coffees and cold beverages. Let \( X = \# \) hot coffees, \( Y = \# \) cold beverages sold per day. Historical data at their Harvard Square café results is the following joint probability distribution \( p(X,Y) \) for \( X \) and \( Y \).

<table>
<thead>
<tr>
<th>Probability ( p_i )</th>
<th>Number of Hot Coffees Sold ( x_j )</th>
<th>Number of Cold Drinks Sold ( y_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>360</td>
<td>360</td>
</tr>
<tr>
<td>0.10</td>
<td>790</td>
<td>110</td>
</tr>
<tr>
<td>0.15</td>
<td>840</td>
<td>30</td>
</tr>
<tr>
<td>0.05</td>
<td>260</td>
<td>90</td>
</tr>
<tr>
<td>0.15</td>
<td>190</td>
<td>450</td>
</tr>
<tr>
<td>0.10</td>
<td>300</td>
<td>230</td>
</tr>
<tr>
<td>0.10</td>
<td>490</td>
<td>60</td>
</tr>
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<td>150</td>
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</tr>
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</tr>
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<td>0.05</td>
<td>510</td>
<td>290</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td><strong>E(X)=457.0</strong></td>
<td><strong>E(Y)=210.0</strong></td>
</tr>
<tr>
<td><strong>Standard Deviation</strong></td>
<td><strong>SD(X)=244.3</strong></td>
<td><strong>SD(Y)=145.6</strong></td>
</tr>
</tbody>
</table>
Comment: It seems that smaller sales of hot coffees are often accompanied by larger sales of cold beverages.
Comment: Hot coffee sales greater than the average number sold per day are typically accompanied by cold beverage sales that are smaller than the average sold per day.
Describing Joint Relationships

**Covariance:**

\[
\text{Cov}(X, Y) = E\left[ (X - \mu_X)(Y - \mu_Y) \right] = \sum_i P(X = x_i, Y = y_i)[(x_i - \mu_X)(y_i - \mu_Y)]
\]

**Correlation:**

\[
\text{CORR}(X, Y) = \frac{\text{COV}(X, Y)}{\sigma_X \sigma_Y}
\]

**Comments . . .**

- The measure of correlation is unit-free.
- \text{CORR}(X, Y) is always between -1.0 and 1.0.
Comment: Hot coffee sales *greater* than the average number sold per day are typically accompanied by cold beverage sales that are *smaller* than the average sold per day.
X = # hot coffees, Y = # cold beverages sold per day

<table>
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<tr>
<td>$p_i$</td>
<td>$x_i$</td>
<td>$y_i$</td>
</tr>
<tr>
<td>0.10</td>
<td>(360, 457)</td>
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<td>(290, 210)</td>
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Mean
- $E(X) = 457.0$
- $E(Y) = 210.0$

Standard Deviation
- $SD(X) = 244.3$
- $SD(Y) = 145.6$

Covariance of $X$ and $Y$

\[
\text{Cov}(X,Y) = 0.10 \times (360 - 457) \times (360 - 210) + 0.10 \times (790 - 457) \times (110 - 210) + \ldots + 0.05 \times (510 - 457) \times (290 - 210) = -23,702
\]

Correlation of $X$ and $Y$

\[
\text{Corr}(X,Y) = \frac{-23,702}{(244.3)(145.6)} = -0.67
\]
Comments . . .

If higher than average values of $X$ are apt to occur with higher than average values of $Y$, then $\text{COV}(X, Y) > 0$ and $\text{CORR}(X, Y) > 0$. i.e., $X$ and $Y$ are positively correlated.

If higher than average values of $X$ are apt to occur with lower than average values of $Y$, then $\text{COV}(X, Y) < 0$ and $\text{CORR}(X, Y) < 0$. i.e., $X$ and $Y$ are negatively correlated.
Linearly related random variables have perfect correlation

If $Y = aX + b$, with $a > 0$, then $\text{Corr}(X, Y) = 1$

If $Y = aX + b$, with $a < 0$, then $\text{Corr}(X, Y) = -1$

“Perfect correlation”!

Examples: $X =$ temperature in F; $Y =$ temperature in C

$X =$ #hot coffees sold in a day;

$Y =$ revenue from hot coffee/day
Caution: Correlation is not the same as Causality!

(See “Cause or Correlation”, The Economist, October 1998.)
Sums of Random Variables

Example: cold beverages are $2.50/glass;
(Y: the number of cold beverages sold in a day)
hot coffees are $1.50/cup.
(X: the number of hot coffees sold in a day)

Find:

- The mean and standard deviation of daily $ sales of cold beverages?
  \[ E(2.5Y) = ? \quad \text{Var}(2.5Y) = ? \]

- The mean and standard deviation of daily $ sales of hot coffees?
  \[ E(1.5X) = ? \quad \text{Var}(1.5X) = ? \]

- The mean and standard deviation of total daily $ sales of both beverages?
  \[ E(1.5X + 2.5Y) = ? \quad \text{Var}(1.5X + 2.5Y) = ? \]
Mean of the Sum of Random Variables

“Expectation of a weighted sum of random variables equals the weighted sum of the expectations”

Rule 1: \[ E(aX + bY) = aE(X) + bE(Y) \]

Variance of the Weighted Sum of Random Variables

Rule 2:

\[ \text{VAR}(aX + bY) = a^2 \text{VAR}(X) + b^2 \text{VAR}(Y) + 2ab \text{COV}(X, Y) \]

or, equivalently:

\[ \text{VAR}(aX + bY) = a^2 \text{VAR}(X) + b^2 \text{VAR}(Y) + 2ab \sigma_X \sigma_Y \text{CORR}(X, Y) \]

If \( X \) and \( Y \) are independent then \( \text{Cov}(X,Y)=0 \) and \( \text{Corr}(X,Y)=0 \)
Comments:

If X and Y are independent, then $\text{VAR}(X + Y) = \text{VAR}(X) + \text{VAR}(Y)$

“Variance of a sum of independent random variables is the sum of the variances”

Note that in particular, these rules imply:

- $\text{E}(aX+b) = a\text{E}(X) + b$
- $\text{VAR}(aX+b) = a^2\text{VAR}(X)$
  - and
- $\text{SD}(aX+b) = |a| \text{SD}(X)$

The constant $b$ shifts the distribution but does not change its shape
- The mean and SD of daily sales of cold beverages?
  \[ E(2.5Y) = 2.5 \times E(Y) = 2.5 \times 210 = \$525, \quad SD(2.5Y) = 2.5 \times 145.6 = \$364 \]

- The mean and SD of daily sales of hot coffees?
  \[ E(1.5X) = 1.5 \times E(X) = 1.5 \times 457 = \$685.5, \quad SD(1.5X) = 1.5 \times 244.3 = \$366.45 \]

- The mean and SD of total daily sales of all beverages?
  \[ E(1.5X + 2.5Y) = 1.5 \times E(X) + 2.5 \times E(Y) = 685.5 + 525 = \$1,210.5 \]

\[
\begin{align*}
\text{Var}(1.5X + 2.5Y) &= (1.5)^2 \times (244.3)^2 + (2.5)^2 \times (145.6)^2 +
2 \times 1.5 \times 2.5 \times 244.3 \times 145.6 \times (-0.6663) \\
&= 89,029
\end{align*}
\]

\[
SD(1.5X + 2.5Y) = \sqrt{89,029} = \$298.38
\]

<table>
<thead>
<tr>
<th>Beverage Type</th>
<th>Mean ((\mu))</th>
<th>Standard Deviation ((\sigma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cold</td>
<td>$525</td>
<td>$364</td>
</tr>
<tr>
<td>Hot</td>
<td>$685.5</td>
<td>$366.45</td>
</tr>
<tr>
<td>Total</td>
<td>$1,210.5</td>
<td>$298.38</td>
</tr>
</tbody>
</table>
Variance Reduction

Notice the surprise in the previous analysis.

The variance and standard deviation of the sum of hot and cold drink sales is less than the variance and standard deviation of either one alone.

This is the principle of diversification to reduce variation and risk.
Asset Diversification

Asset diversification is the cornerstone of modern finance.

The simple analysis that follows extends to yield the CAPM (Capital Asset Pricing Model) and portfolio models and portfolio theory. This is a key business concept.

The Nobel Prize in economics (1990) was awarded to Merton Miller, William Sharpe and Harry Markowitz for their work on portfolio theory and portfolio models (and the implications for asset pricing).
**Expected Return and Risk in Investing**

- The future rate of return on investment in an asset is a random variable.
- Other things being equal, investors prefer higher expected return.
- Other things being equal, investors prefer lower risk.
- We measure risk with the standard deviation of the rate of return.

<table>
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<tr>
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<td>----</td>
<td>0.41</td>
<td>0.13</td>
<td></td>
</tr>
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<td>9.1</td>
<td>16.5</td>
<td>0.41</td>
<td>----</td>
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- Given the choice to invest all money in one of these three, rational investors would not choose Boeing: it has lower expected return and more risk.
- **Most of us are risk-averse about certain investment decisions, some more than others (as we have seen in various decision examples).**
Diversified Portfolio

Let $X$ (for Boeing) and $Y$ (for GM) be the r.v.s that denote the annual return (in %) on these two assets. Suppose we have reliable estimates for $m_X$, $m_Y$, $s_X$, $s_Y$, and $\text{Corr}(X,Y)$ as given in the table.

Suppose that we invest a fraction $f$ in Boeing and a fraction $1-f$ in GM. Let $W = \text{annual return on our diversified portfolio of the two assets.}$

Then,

$$W = f X + (1-f) Y$$

$$m_W = E(W) = f E(X) + (1 - f) E(Y)$$

$$m_W = E(W) = 9.1 f + 12.1 (1-f)$$

$$\mu_X = 9.1, \mu_Y = 12.1,$$

$$\sigma_X = 16.5, \sigma_Y = 15.8 \quad \text{(in %)}$$

$$\text{Corr}(X,Y) = -0.22$$

$$\text{Var}(W) = f^2 \text{Var}(X) + (1-f)^2 \text{Var}(Y) + 2 f (1-f) s_X s_Y \text{Corr}(X,Y)$$

$$\text{Var}(W) = 16.5^2 f^2 + 15.8^2 (1-f)^2 + 2 f (1-f) (16.5)(15.8)(-0.22)$$

$$\sigma_W = \text{SD}(W) = \sqrt{272.25 f^2 + 249.64 (1-f)^2 + 2 f (1-f) (16.5)(15.8)(-0.22)}$$
\[ \mu_X = 9.1, \mu_Y = 12.1 \]
\[ \sigma_X = 16.5, \quad \sigma_Y = 15.8 \]
Risk and Expected Return

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<th>Risk (Standard Deviation) (%)</th>
<th>Expected Return (%)</th>
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The graph illustrates the relationship between risk (standard deviation) and expected return for three companies: Boeing, GM, and Exxon. The x-axis represents the risk (standard deviation) in percentage, while the y-axis represents the expected return in percentage.
Effect of Correlation on Portfolio Risk Levels

- CORR = -1.0
- CORR = -0.5
- CORR = 0.0
- CORR = 0.5
- CORR = 1.0

All GM
All Boeing
Examples of Diversification

- Stock portfolios
- Pharmaceutical companies managing multiple drugs with expected returns and risks
- Pratt & Whitney commercial and military divisions: 1990s commercial is the profit engine, but post 9/11…
- Others?
Many companies sell software packages for asset diversification, also known as “portfolio optimization.”

- BARRA (www.barra.com)  
  Aegis System-Optimizer
- Wilson Associates International (www.wilsonintl.com)  
  Power Optimizer  
  RAMCAP  
  Xpress
- LaPorte Asset Allocation System (www.laportesoft.com)

Typical features of these systems include:
* Historical databases
* Graphical capabilities
* Reporting capabilities
* Technical support

Typical prices are $2,000 to $10,000 for an initial license plus $1,000 to $4,000 for software and/or database upgrades.
Portfolio Risk Management is one of the most important financial concepts of the last two decades, underlying all kinds of investment decisions. Yet it is based on very simple statistical principles. In Lecture 9, we look at the ubiquitous Normal distribution, adding some other very useful ideas and tools.