15.401 Recitation

Extra Session: Mid-Term Review
General Advice

• Show your work! Answers only give you partial credit
• Write down the formulas you use
• Draw timelines for cash flows
• Make sure you apply the annuity/ perpetuity formulas correctly – Example:

\[ PV \text{ (Annuity)} = A \times \frac{1}{r} \left[1 - \frac{1}{(1+r)^T}\right] \]

• State your assumptions
• Leave plenty of decimal places for interest rates (e.g., 1.2345%)
Sample Midterm Solutions

- Q – Compounding
- Q – APR-EAR Conversion
- Q – Common
Q1 – Compounding

- If the annual interest rate is 10 percent, how long would you have to wait before a $17,500 investment doubles in value?
Q1 – Compounding

• If the annual interest rate is 10 percent, how long would you have to wait before a $17,500 investment doubles in value?

Advice:
• Question whether the rate is expressed as EAR, APR, or other; and what is the compounding period
• If some of the above is not clear to you, state your assumptions
• Write down the equation in terms of $T$
• Compute precise value, then round up
• Show your work!
Q1 – Compounding

- Let $T$ be the amount of time (in years) required, then

$$17,500 \cdot (1+10\%)^T = 17,500 \cdot 2$$

$$1.1^T = 2$$

$$T \cdot \ln(1.1) = \ln(2)$$

$$T = \frac{\ln 2}{\ln 1.1} = 7.2725$$

- The minimum number of years is 8.
Q2 – APR-EAR Conversion

- Your car dealer offers you a loan for part of the purchase price of a new car, citing an annual percentage rate (APR) of 8.5%. What is the effective annual rate of such a loan (recall that an auto loan typically requires monthly payments)?
Q2 – APR-EAR Conversion

- Your car dealer offers you a loan for part of the purchase price of a new car, citing an annual percentage rate (APR) of 8.5%. What is the effective annual rate of such a loan (recall that an auto loan typically requires monthly payments)?

Advice:

- Know how to EAR ↔ APR

\[
\text{Effective Rate per Compounding Interval} = \frac{r_{\text{APR}}}{k} \quad r_{\text{EAR}} = \left(1 + \frac{r_{\text{APR}}}{k}\right)^k - 1.
\]

- Leave at least 2 decimal places in the end
Q2 – APR-EAR Conversion

• Given monthly-compounded APR, we have

\[
\begin{align*}
    r_{EAR} &= \left(1 + \frac{r_{APR}}{k}\right)^k - 1 \\
    &= \left(1 + \frac{8.5\%}{12}\right)^{12} - 1 \\
    &= 8.8391\%
\end{align*}
\]
Q3 – Common Stocks

- Company ABC has just paid a dividend of 50 cents per share. Because of its growth potential, its dividend is forecasted to grow at a rate of 7 percent per year indefinitely. If the company's appropriate cost of capital (given its risk) is 11 percent, what was ABC's share price immediately before it paid its 50 cent dividend, i.e., the stock price right before the ex-dividend date?
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Advice:
- Be careful with the times when dividend payment occurs – draw a timeline!
- In this case, remember to include D₀
- Write the growing perpetuity formula
- Use the formula correctly – numerator is D₁, not D₀
Q3 – Common Stocks

- Parameters:
  \[ D_0 = 0.50; \ D_1 = 0.50 \times 1.07; \ g = 0.07; \ r = 0.11 \]

  \[ D_0 = D_0; \quad D_1 = D_0(1+g); \quad D_2 = D_0(1+g)^2; \ldots \]

- Dividend discount model:
  \[
P_0 = D_0 + \frac{D_1}{r - g}
  = 0.50 + \frac{0.50 \times 1.07}{0.11 - 0.07}
  = $13.88
  \]
Q3 – Common Stocks

- **Parameters:**
  \[ D_0 = 0.50; \quad D_1 = 0.50 \times 1.07; \quad g = 0.07; \quad r = 0.11 \]

- **Dividend discount model:**

  \[
  P_0 = D_0 + \frac{D_1}{r - g}
  \]

  \[
  = 0.50 + \frac{0.50 \times 1.07}{0.11 - 0.07}
  \]

  \[
  = $13.88
  \]
Q4 – Fixed Income Securities

- The current market price of a two-year 25 percent coupon bond with a $1,000 face value is $1,219.71 (recall that such a bond pays coupons of $250 at the end of years 1 and 2, and the principal of $1,000 at the end of year 2). The current market price of a one-year pure discount bond with a $50 face value is $44.64.

a) What must the price of a two-year pure discount bond with a $2,500 face value be in order to avoid arbitrage?

Advice:
- Do not confuse $r_1$ with $r_2$ and YTM
- In this case, do not discount the coupon with the YTM
Q4 – Fixed Income Securities

- “The current market price of a one-year pure discount bond with a $50 face value is $44.64”:

\[
\frac{50}{(1 + r_1)} = 44.64 \Rightarrow r_1 = 12.0072\%
\]

- “The current market price of a two-year 25 percent coupon bond with a $1,000 face value is $1,219.71”:

\[
\frac{250}{(1 + r_1)} + \frac{1,250}{(1 + r_2)^2} = 1,279.71
\]

\[
\frac{250}{(1 + 0.120072)} + \frac{1,250}{(1 + r_2)^2} = 1,279.71 \Rightarrow r_2 = 11.9990\%
\]

- Now we know the spot rates, \( r_1 \) and \( r_2 \)
Q4 – Fixed Income Securities

- The price of the two-year pure discount bond must be

\[ P = \frac{2,500}{(1 + r_2)^2} = \frac{2,500}{(1 + 0.119990)^2} = $1,993.02 \]
Q4 – Fixed Income Securities

• The price of the two-year pure discount bond must be

\[ P = \frac{2,500}{(1 + r_2)^2} = \frac{2,500}{(1 + 0.119990)^2} = $1,993.02 \]

b) Suppose the price of the two-year pure discount bond with a $2,500 face value is only $1,900. Is there an arbitrage opportunity? Is yes, how would you structure a trade that has zero cash flow in years 1 and 2 and a positive cash flow only in year 0 (i.e. now).
Q4 – Fixed Income Securities

• The price of the two-year pure discount bond must be

\[ P = \frac{2,500}{(1+r_2)^2} = \frac{2,500}{(1+0.119990)^2} = \$1,993.02 \]

b) Suppose the price of the two-year pure discount bond with a $2,500 face value is only $1,900. Is there an arbitrage opportunity? Is yes, how would you structure a trade that has zero cash flow in years 1 and 2 and a positive cash flow only in year 0 (i.e. now).

• There must be an arbitrage opportunity because $1,900 is not the fair price of $1,993.02! How to capitalize it?
Q4 – Fixed Income Securities

Advice:
- Consider all the instruments you may use – 3 in this example
- Build a table that shows for every year that are expected to come from each bond (see below)
- In this case, start by writing the cash flows for the bond that is not fairly priced, or the one with longer maturity
- Use your logic!

<table>
<thead>
<tr>
<th>Bond</th>
<th>Position</th>
<th>CF at 0</th>
<th>CF at 1</th>
<th>CF at 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-yr, Zero Coupon, $50 Par,</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sells for $44.64</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-year, 25% coupon, $1,000 Par, sells for</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>$1,219.71</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1,900</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOTAL:</td>
<td>+ ?</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Q4 – Fixed Income Securities

Advice:

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<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-yr, Zero Coupon, $2,500 Par, sells for $1,900</td>
<td>Buy (Long) 1 bond</td>
<td>-1,900</td>
<td></td>
<td>2,500</td>
</tr>
</tbody>
</table>

TOTAL: + ? 0 0
**Q4 – Fixed Income Securities**

**Advice:**
- Consider all the instruments you may use – 3 in this example
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-year, 25% coupon, $1,000 Par,</td>
<td>Sell (Short)</td>
<td>+2,439.42</td>
<td>-500</td>
<td>-2,500</td>
</tr>
<tr>
<td>Par, sells for $1,219.71</td>
<td>2 bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-yr, Zero Coupon, $2,500 Par,</td>
<td>Buy (Long)</td>
<td>-1,900</td>
<td></td>
<td>2,500</td>
</tr>
<tr>
<td>Par, sells for $1,900</td>
<td>1 bond</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>TOTAL:</strong></td>
<td>+ ?</td>
<td>0</td>
<td>0</td>
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Q4 – Fixed Income Securities

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<th>CF at 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-yr, Zero Coupon, $50 Par, sells for $44.64</td>
<td>Buy (Long) 10 bonds</td>
<td>-446.40</td>
<td>+500</td>
<td></td>
</tr>
<tr>
<td>2-year, 25% coupon, $1,000 Par, sells for $1,219.71</td>
<td>Sell (Short) 2 bonds</td>
<td>+2,439.42</td>
<td>-500</td>
<td>-2,500</td>
</tr>
<tr>
<td>2-yr, Zero Coupon, $2,500 Par, sells for $1,900</td>
<td>Buy (Long) 1 bond</td>
<td>-1,900</td>
<td>2,500</td>
<td></td>
</tr>
<tr>
<td><strong>TOTAL:</strong></td>
<td></td>
<td><strong>+ 93.02</strong></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Q4 – Fixed Income Securities

<table>
<thead>
<tr>
<th>Bond</th>
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<th>CF at 0</th>
<th>CF at 1</th>
<th>CF at 2</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Buy (Long)</td>
<td>-446.40</td>
<td>+500</td>
<td></td>
</tr>
<tr>
<td>sells for $44.64</td>
<td>10 bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-year, 25% coupon, $1,000 Par,</td>
<td>Sell (Short)</td>
<td>+2,439.42</td>
<td>-500</td>
<td>-2,500</td>
</tr>
<tr>
<td>sells for $1,219.71</td>
<td>2 bonds</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2-yr, Zero Coupon, $2,500 Par,</td>
<td>Buy (Long)</td>
<td>-1,900</td>
<td></td>
<td>2,500</td>
</tr>
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<td></td>
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<tr>
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<td></td>
<td>+ 93.02</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Note:
- We can make free money today, with no risk – arbitrage!
- In our example the profit is $93.02, the same amount by which each 2-year, zero coupon, is underpriced
- There is more than solution to this problem, but all of them are multiples of this simple case
Q4 – Fixed Income Securities

Alternative method:

• Solve and equation system:

<table>
<thead>
<tr>
<th></th>
<th>CF at 0</th>
<th>CF at 1</th>
<th>CF at 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1 \times B_1$</td>
<td>$-n_1 \times 1219.71$</td>
<td>$n_1 \times 250$</td>
<td>$n_1 \times 1250$</td>
</tr>
<tr>
<td>$n_2 \times B_2$</td>
<td>$-n_2 \times 44.64$</td>
<td>$n_2 \times 50$</td>
<td>$0$</td>
</tr>
<tr>
<td>$n_3 \times B_3$</td>
<td>$-n_3 \times 1900$</td>
<td>$0$</td>
<td>$n_3 \times 2500$</td>
</tr>
<tr>
<td>$+$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Q4 – Fixed Income Securities

• This portfolio is an arbitrage portfolio if

\[
\begin{align*}
    w_1 \times 1250 + w_3 \times 2500 &= 0 \\
    w_1 \times 250 + w_2 \times 50 &= 0 \\
    -w_1 \times 1219.71 - w_2 \times 44.64 - w_3 \times 1900 &> 0
\end{align*}
\]

• Solution of the system:

\[
\begin{align*}
    w_1 &= -2k \\
    w_2 &= 10k \quad \text{for any } k > 0 \\
    w_3 &= k
\end{align*}
\]
Q5 – Present Value

- Your friend is celebrating her 35th birthday today and wants to start saving for her anticipated retirement at age 65 (she will retire on her 65th birthday). She would like to be able to withdraw $80,000 from her savings account on each birthday for at least 20 years following her retirement (the first withdrawal will be on her 66th birthday). Your friend intends to invest her money in the local savings bank which offers 4% per year. She wants to make equal annual deposits on each birthday in a new savings account she will establish for her retirement fund.

If she starts making these deposits on her 36th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals upon retirement?
Q5 – Present Value

Advice:

- Draw a timeline!
- Shift the timeline so that now is $t = 0$
- Write down the annuity formula
- Find out the correct discount rate for the cash flows, in this case 4%
Q5 – Present Value

Advice:
- Draw a timeline!
- Shift the timeline so that now is $t = 0$
- Write down the annuity formula
- Find out the correct discount rate for the cash flows, in this case 4%

![Timeline Diagram]

Age: 35 36 37
Time: 0 1 2

80k 80k 80k
Q5 – Present Value

- Let “A” be the necessary annual deposit, then, using the annuity formulas:

\[
PV(\text{withdraws}) = PV(\text{savings})
\]

\[
\frac{1}{(1.04)^{30}} \cdot \left[ \frac{80,000}{0.04} \left( 1 - \frac{1}{1.04^{30}} \right) \right] = \frac{A}{0.04} \left( 1 - \frac{1}{1.04^{30}} \right)
\]

\[
335,212.11 = A \cdot 17.2920
\]

\[
A = $19,385.35
\]
Q5 – Present Value

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\[ PV(\text{withdraws}) = PV(\text{savings}) \]

\[
\frac{1}{(1.04)^{30}} \cdot \left[ \frac{80,000 \cdot \left( 1 - \frac{1}{1.04^{20}} \right)}{0.04} \right] = \frac{A}{0.04} \left( 1 - \frac{1}{1.04^{30}} \right)
\]

\[
335,212,11 = A \cdot 19.3854
\]

\[
A = \$19,385.35
\]

- Note that the $80K annuity has been discounted 30 years
Q6 – Forwards & Futures

• The current level of the S&P 500 is $1040. The risk-free interest rate per year is 2%. Assume negligible dividends. The 6 month futures contract is trading at $1060.

a) Is there an arbitrage opportunity? Briefly explain.

Advice:

• Write down the spot-futures parity
• Make sure that “T”, “r_f” and “γ” are based on the same unit of time
Q6 – Forwards & Futures

- Spot-futures parity requires

\[ F_T = S_0 (1 + r - y)^T \]

\[ 1040 (1 + 2\%)^{0.5} = 1050.35 \neq 1060 \]

- There is an arbitrage opportunity because the trading price is not equal to the “fair price” (calculated with the spot-futures parity)

b) If there is an arbitrage opportunity, what strategy would you use to exploit it without using any funds of your own?
Q6 – Forwards & Futures

Advice:

- Draw the CF table – similar to that in question 4, with all the instruments you have
- Note that you can always borrow or lend money at the risk free rate – eg. \(1,040 \times (1+0.02)^{0.5} = 1,050.35\)
- Use your logic – And if you arrive to the inverse conclusion just inverse all the signs in the table

<table>
<thead>
<tr>
<th>Security</th>
<th>Position</th>
<th>CF at 0</th>
<th>CF at 6 months</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500 stock trading at $1,040</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Borrow / Lend Money</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 month future trading at $1,060</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>TOTAL:</strong></td>
<td></td>
<td></td>
<td></td>
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Q6 – Forwards & Futures

Advice:
- Draw the CF table – similar to that in question 4, with all the instruments you have
- Note that you can always borrow or lend money at the risk free rate – eg. $1,040*(1+0.02)^{0.5} = $1,050.35
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<tr>
<td>6 month future trading at $1,060</td>
<td>Short 1</td>
<td>0</td>
<td>$1,060 – S_T</td>
</tr>
<tr>
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<td></td>
<td></td>
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Q6 – Forwards & Futures

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<tr>
<td><strong>TOTAL:</strong></td>
<td></td>
<td><strong>$0</strong></td>
<td><strong>$9.65</strong></td>
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Q6 – Forwards & Futures

Note:
- Arbitrage! In 6 months, we will have made $9.65 without any risk or investment

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<td>-1,050.35</td>
</tr>
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<td>6 month future trading at $1,060</td>
<td>Short 1</td>
<td>0</td>
<td>1,060 – $S_T</td>
</tr>
<tr>
<td>TOTAL:</td>
<td>$0</td>
<td>$9.65</td>
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