Numerical Methods for ODE

A) Reduction to 1st order system

B) Discretization

C) Stability

Reading: Numerical Comp. of Int. & Ext. Flows Vol. I C. Hirsch 267-290

An $N^{th}$ order ODE can always be reduced to $N$ 1st order ODEs:

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

Define:

$$x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}$$

$$\Rightarrow$$

$$x_1' = x_2$$

$$x_2' = x_3$$

$$\vdots$$

$$x_{n-1}' = x_n$$

$$x_n' = F(t, x_1, x_2, \ldots, x_n)$$

Example: Falkner-Skan Eqn for $F(y)$

$$F'''' + \frac{\beta u + 1}{2} F''' + \beta u (1 - F'^2) = 0$$

$$F' = 0$$

$$U' = S$$

$$S' = -\frac{\beta u + 1}{2} F S - \beta u (1 - U^2)$$

where

$$\left[ \begin{array}{c} F' \\ U' \\ S' \end{array} \right] = f \left[ \begin{array}{c} F \\ U \\ S \end{array} \right]; \beta$$
Beam Equation

\[ (EI(x)w'')'' = p(x) \]

1. \[ w' = t \]
   \[ t' = u \]
   \[ u' = v \]

\[ EIv' + 2EIv + EI''u = p \]

\[ [EIu]' = [(EI)'u + v''EI]' \]

2. Alternatively

\[ w' = t \]
\[ EIt' = u \]
\[ u' = v \]
\[ v' = p \]

We need to examine how to solve 1st order ODEs for \( y'(x, y) \).

Discretization

1. \( y(x) \) governed by \( ODE \) and ICs and/or BCs
2. \( \infty \) DOF

Discrete System \( i = 1 \leq N \)

\( y_i \) governed by \( N \) algebraic equations (including IC, BC)

\( N \) DOFs
Example

\[ y' = -xy : \alpha > 0 \]

Exact solution: \[ y = y_0 e^{-\alpha x} \]

Discretize using forward Euler

\[ y_{i+1} = y_i + \Delta x y_i' = y_i - \Delta x \alpha y_i = y_i (1 - \Delta x \alpha) \]

\[ y_{i+2} = y_{i+1} - \Delta x \alpha y_{i+1} = y_{i+1} (1 - \Delta x \alpha) = y_i (1 - \Delta x \alpha)^2 \]

In general: \[ y_n = y_0 (1 - \alpha \Delta x)^n \]

The discretization is \underline{consistent} if \[ y_n \to y_{\text{exact}} \text{ as } \Delta x \to 0 \]

\[ \lim_{n \to \infty} y_n (1 - \alpha \Delta x)^n, \text{ where } X = \text{end} - x_0, \text{ length of domain} \]

\[ = y_0 e^{-\alpha x} \]

\underline{Discretization is consistent}

\underline{Stability}

Discretization is \underline{stable} if error between \( y_i \) & \( y_{\text{exact}} \)

stays bound as \( n \to \infty \).

\[ y_n = y_0 (1 - \alpha \Delta x)^n \]

\[ \left| \frac{y_n}{y_0} \right| = (1 - \alpha \Delta x)^n, \quad \alpha > 0 \]

for \( \alpha > 0 \) \( y_{\text{exact}} = y_0 e^{-\alpha x} \) decays as \( x \to \infty \)

\[ \text{for } \left| \frac{y_n}{y_0} \right| \text{ to decay } \left| 1 - \alpha \Delta x \right| < 1 \quad \left( \left| 1 - \alpha \Delta x \right| > 1, \text{ grows} \right) \]

\[ \left| 1 - \alpha \Delta x \right| < 1 \text{ is a stability requirement} \]
\[ x \Delta x < 2 \quad \text{for stability} \]

\[ y = -\alpha y \]

\[ y_{i+1} = y_i + \Delta x \frac{y_{i+1}}{1 + \alpha \Delta x} \]

\[ y_n = y_0 (1 + \alpha \Delta x)^{-n} \]

\[ \alpha > 0 \quad \text{stable if} \quad |1 + \alpha \Delta x| \geq 1 \quad \Rightarrow \quad \text{stable for all } \Delta x \]

For simple 1-equation systems, stability and accuracy requirements on \( \Delta x \) are typically the same.

For multi-equation systems, they can be very different.

\[ \epsilon y'' + y' - y = 0 \quad y(0) = 1 \]
\[ y'(0) = 0 \]

Reduce order to 1st

\[ z = y' - y \]

\[ y' = -y + z \quad y(0) = 1 \]
\[ z(0) = 1 \]

Exact solution
\[ \ddot{y} = -[A] \dot{y} \]

\[ y_{n+1} = (I - [A] \Delta x) y_n \]

\[ [A] \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

if the largest eigenvalue of \([A]\) has magnitude less than 1. Then \(y_n\) is bounded, or

\[ \Delta x \leq \frac{2}{\lambda_{\max}} \]
Using Forward Euler:

\[ y_{i+1} = y_i + \Delta x (-y_i + z_i) = y_i (1 - \Delta x) + z_i \Delta x \]

\[ z_{i+1} = z_i - \frac{z_i}{\epsilon} \Delta x = z_i (1 - \frac{\Delta x}{\epsilon}) \]

\[ \begin{bmatrix} \Delta x y_i \\ \Delta x z_i \end{bmatrix} = \begin{bmatrix} 1 - \Delta x & \Delta x \\ 0 & (1 - \frac{\Delta x}{\epsilon}) \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} \]

To accurately resolve \( z \) close to \( x = 0 \), we need \( \Delta x = o(\epsilon) \), and \( \Delta x \ll o(1) \) away from \( x = 0 \).

\( z_i \) may go unstable depending on \( y_i \). Accuracy may result in instability even after \( z \rightarrow 0 \). More work is done in maintaining small \( \Delta x = o(\epsilon) \) (exam work \( \approx \frac{1}{\epsilon} \)).

Stability in \( z_i \) contaminates \( y_i \). Alternative is to use Backward Euler Scheme:

\[ \Delta x \text{stability} \ll \Delta x \text{accuracy} \rightarrow \text{stiff ODE system} \]

Stiffness occurs when there are two or more different scales of the independent variable:

\[ \begin{align*}
y' &= z - y \\
y'' &= z' - y' \\
\epsilon z' &= -z \\
\epsilon (y'' + y') &= -y' + y \\
\epsilon y'' + y' &= \]

Stability

\[ y_{n+1} = y_n (1 - \alpha \Delta x) \]
\[ y_n = y_0 (1 - \alpha \Delta x)^n \]

Error

\[ y_n = y_{\text{exact}} + \epsilon_n \]

Substituting above gives

\[ \epsilon_{n+1} = \epsilon_n (1 - \alpha \Delta x) \quad (\text{since } y_{\text{exact}} \text{ satisfies}) \]

For the forward Euler scheme to be stable
\[ |\frac{\epsilon_{n+1}}{\epsilon_n}| \text{ must not grow } = |1| \]

Assume

\[ \epsilon_n = y_n e^{i \phi} \]
\[ y_{n+1} e^{i (n+1) \phi} = y_n e^{i \phi} (1 - \alpha \Delta x) \]
\[ \frac{y_{n+1}}{y_n} = \frac{(1 - \alpha \Delta x) e^{-i \phi}}{2 \text{ amplification factor}} \]
\[ |\frac{y_{n+1}}{y_n}|^2 = (1 - \alpha \Delta x)^2 \]
\[ \Rightarrow |1 - \alpha \Delta x| \leq 1 \implies \alpha \Delta x < 2 \]

Stiff system

\[ ey'' + y' (1 + e) + y = 0 \quad y(0) = 1 \]
\[ z' = y' + y, z'' = y'' + y' \]

\[ \Rightarrow e(y'' + y') + (y' + y) = 0 \]
\[ e z' + z = 0 \implies z' = -\frac{z}{e} \]

Equivalent system:

\[ y' = -y + z \]
\[ z' = -\frac{z}{e} \]

Stability on \( z \)
\[ |\frac{\Delta x}{e}| < 2 \implies \Delta x < 2e \]