7.3) Integral Methods for Turbulent Flow

a) Two term momentum relation
b) Rayleigh relation

Recall

\[ \frac{G^2}{A^2} = 1 + 8 \beta \]

which gave us quantitative relationship between \( G, \frac{dG}{dx}, H \)

Recall IBL eqns:

\[ \frac{dG}{dx} = \frac{G}{L} - (H+2) \frac{dG}{dx} \]

\[ \frac{dH^*}{dx} = \frac{2C_0}{H^*} - \frac{G}{L} + (H-1) \frac{dG}{dx} \]

\[ C_0 = \frac{1}{\rho U_c^3} \int_{y_0}^{y} \left( \frac{\partial p}{\partial y} \right) \; dy \]

\[ \int_{y_0}^{y} \left[ \mu \left( \frac{\partial u}{\partial y} \right)^2 + (-\rho U_c v) \frac{\partial u}{\partial y} \right] dy \]

We require \( G, C_0, H^* \) closure relationship.

\( 0 \) Turbulent Case: We are given

\[ \frac{u_*}{U_c} = f \left( \frac{U_c y^*}{U_*}, \beta \right) \]

\[ \frac{U_c}{y^*} = g \left( y^*, \beta \right) \]
\[ u^+ = \frac{\sqrt{2}}{g} \phi(y^{1/3}, \beta) \quad \text{turb} \]

\[ u^+ = g(\delta^+) + c(\beta) \]

\[ = g(\text{Re}_x \sqrt{g/\beta}) + c(\beta) \]

\[ \frac{H^*}{u^*} = \frac{\sqrt{2}}{g} \phi^* \quad \text{Kuss, } \beta \]

\[ \phi^* = \phi(\text{Re}_x, \text{Nu}) \quad \text{functional form, obtain expression from curve fit from different values of } H^*, \text{Re}_x \text{, see handout eqn (2.11)} \]

**H**

Laminar \( H^* = H^*(H) \)

Turbulent \( H^* = H^*(H; \text{Re}_x) \)

\[ \Rightarrow \quad \text{Co} = \frac{C}{C_H(\text{Re}_x)} \quad \text{and} \quad H^* = H^*(H, \text{Re}_o) \]

\[ \text{Co} = \frac{1}{\mu_c} \int_0^\infty \frac{\partial u}{\partial y} \, dy \]

Can be separated into inner and outer contribution
\[
C_{Dc} = \frac{1}{2} \rho u_e^3 \int_0^{\frac{U_e}{m}} \frac{du}{g} = \frac{C_1}{\varepsilon} \cdot U_e
\]

\[
C_{Dc} = \frac{1}{2} \rho u_e^3 \int_0^{\frac{U_e}{m}} \frac{du}{g} - dy
\]

\[
C_{Dc} = \frac{1}{2} \rho u_e^3 \cdot \frac{\pi}{4} \cdot (U_e - U_s)
\]

Combining these, we get

\[
C_0 = \frac{G}{2} \cdot \frac{U_e}{m} + \frac{\pi}{4} \cdot C_1 (1 - U_s)
\]

We can also arrive at this result using the concept of equilibrium flow:

\[
G = \text{const.}, \quad \frac{dG}{dx} = \text{small} \quad \text{for turbulent flow (slow change)}
\]

\[
\frac{dG}{dx} = 0 \quad \Rightarrow \quad \frac{dH}{dx} = 0
\]

\[
\frac{dH}{dx} = 0 \quad \text{weakly depend}
\]

\[
\frac{dH^*}{dx} = 0
\]
Therefore from K.E. eqn.

\[
\frac{\partial}{\partial x} \frac{\dot{H}}{H^2} = \frac{2C_0}{H^2} - \frac{g}{2} \frac{\partial u}{\partial x} + (H-1) \frac{\partial \mu_c}{\partial x}
\]

\[
\Rightarrow \frac{2C_0}{H^2} = \frac{g}{2} \left[ 1 + (\frac{H-1}{H}) \frac{\partial u}{\partial x} \right]
\]

Using \(C-\beta\) equation

\[
\frac{2C_0}{H^2} = \frac{g}{2} \left[ 1 + (\frac{H-1}{H}) \left( \frac{G^2}{A_b} \right) }{ - \frac{1}{B} \right]
\]

\[
= \frac{g}{2} \left[ 1 + (\frac{H-1}{H}) \frac{1}{0.75} \right] \left( \frac{H-1}{H} \right)^{0.02} \left( \frac{H-1}{H} \right)^{0.02} \left( \frac{H-1}{H} \right)^{0.02}
\]

Recall

\[
\frac{H-1}{H} = \frac{3}{2} (1 - \nu)
\]

\[
\frac{2C_0}{H^2} = \frac{g}{2} \cdot V_s + 0.03 \left( \frac{H-1}{H} \right)^2 \frac{3}{4} (1 - \nu)
\]

\[\text{streamwise} \quad \text{velocity} \quad \text{streamwise} \quad \text{velocity}
\]

\[\uparrow \Delta u \quad \uparrow \Delta u\]

\[\text{factor} \rightarrow F C_0 (1 - \nu)\]
Non-equilibrium effects. 

Some $C_e$ is valid for non-equilibrium flows. 

$C_e$ - distinction of $K_e$ into $K$ and $H$. 

$$C_e = \frac{\pi}{4} K (1-U_s)^2$$

$C_e$ is valid for local dependence on $H$. 

Straight except. 

$$\frac{\partial C_e}{\partial x} = U \partial U$$

$C_e > C_{eq}$ 

$C_e < C_{eq}$

$\Rightarrow C_e$ is different 

Lag effect: $\overline{U'}$ depends not only on local $Re$, $M$, but also on upstream $BL$ evolution (history). 

$$\frac{\partial C_e}{\partial x} = \frac{1}{2} U_s + C_e(x) (1-U_s)$$

$\uparrow$ 

no lag 

$\uparrow$ 

lag effect.
\( C_t(x) \) is an independent variable. Introduce

3rd equation:

\[
\frac{dC_t}{dx} = \ldots
\]

\[
\frac{\delta}{\alpha} \frac{dC_t}{dx} = 4.2 \left( \sqrt{C_{eq}} - \sqrt{C_t} \right)
\]

Deflection of leading edge from G-\( \beta \) locus is governed by lag equation.