30 Boundary Layer

A) Crossflow Instability

B) Transition Mechanism

Reading: Whitel 342 - 344, Sami and Reed Annual Review 1989

\( \text{Stability of 3-D BLs} \)

A) 3D O-S Equations:

Recall, for mean flow we can add small perturbation, linearize and derive O-S equations:

\[ \nabla \cdot \vec{u} = 0 \]

\[ \frac{Du}{Dt} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u} \]

\[ \vec{u} = \vec{U} + \vec{u} \]

\[ \rho = \rho_0 + \hat{\rho} \]

Subst. and neglecting higher powers give linearized PDE for disturbance

Assume disturbance is of the form

\[ \vec{u} = \vec{u}(y) e^{i(\alpha x + \beta z - \omega t)} \]

\[ \hat{\rho} = \hat{\rho}(y) e^{i(\alpha x + \beta z - \omega t)} \]

General wave number vector \( \vec{\alpha} + \vec{\beta} \)

\[ \frac{\partial^2 \vec{u}}{\partial x^2} + \frac{\alpha^2}{\rho_0} \vec{u} = 0 \]

\[ -i \omega \vec{u} + i \alpha \vec{U} + i \beta \vec{W} + U \vec{v} = -i \frac{\alpha \hat{\rho}}{\rho} + \frac{1}{Re} \left[ \frac{\partial^2 - \alpha^2 - \beta^2}{\rho_0} \vec{u} \right] \]
\(- i \omega \tilde{v} + i \omega \tilde{v} U + i \beta \tilde{w} U = - \frac{1}{\rho} \frac{d}{dy} \tilde{p} + \frac{1}{\text{Re}} \left[ \omega^2 - \alpha^2 - \beta^2 \right] \tilde{v} \)

\[- i \omega \tilde{w} + i \alpha \tilde{w} U + i \beta \tilde{w} W + \frac{a \omega}{\text{Re}} \frac{d}{dy} \tilde{w} = - \frac{i}{\rho} \tilde{p} \tilde{p} + \frac{i}{\text{Re}} \left[ \omega^2 - \alpha^2 - \beta^2 \right] \tilde{w} \]

\[- i \omega (\alpha \tilde{u} + \beta \tilde{w}) + i \alpha \tilde{u} U + i \alpha \tilde{p} \tilde{w} + \alpha U \tilde{v} \]

\[+ i \alpha \tilde{p} \tilde{w} U + i \beta \tilde{p} \tilde{w} W + \beta W \tilde{v} \]

\[- i \omega \tilde{u} + i (\alpha \tilde{u} U + \beta \tilde{w} W) \]

Add \(\omega (1) + \rho (3) \Rightarrow \) we get:

\[i \alpha' \tilde{w} + \tilde{v} = 0 \]

\[i \alpha' \tilde{u}' (U' - c) + U' \tilde{v} = - \frac{i}{\rho} \alpha' \tilde{p} \]

\[+ \frac{1}{\text{Re}} \left[ \omega^2 - \alpha'^2 \right] \tilde{u}' \]

\[i \alpha \tilde{v} (U' - c) = - \frac{i}{\rho} \frac{d}{dy} \tilde{p} + \frac{1}{\text{Re}} \left[ \omega^2 - \alpha^2 \right] \tilde{v} \]

Do Eqs. with \(\tilde{u}', \tilde{v}', \tilde{u}, \tilde{v} \) as wave vector direction

Do analysis for effective disp. along \(\phi \)

Inputs: \(U(y), W(y), \rho \) (eliminate pressure)

Eigenvalue Prods:
- \(\omega \) if \(\alpha, \beta \) specified (\((x, y)\) spatial)
- \(\alpha \) if \(\rho, \omega \) specified (\((x, \phi)\) spatial)
- \(\beta \) if \(\alpha, \omega \) specified (\((y, \phi)\) spatial)
2D result is $W = 0$.

Assume that $x$-aligned with $\vec{q}$

$$U(y_e) = u_e, \quad W(y_e) = 0$$

in the framework of the approximate approach discussed earlier.

Transition region: Intermittent instability:

- $\alpha_i < 0$ (T-S waves)
- $\beta_i < 0$ (crossflow waves)

Calculation Results:

For

$$\frac{U_e S}{2} > Re_{crit}, \quad \alpha_i < 0$$

promoted by inflection $U(y)$

(advane of only)

For

$$\frac{W_{max} S}{2} > Re_{crit}, \quad \beta < 0$$

promoted by inflection $W(y)$

(always!!)

Example: Johnston

$$W = \begin{cases} \frac{u \tan \beta}{1 - U \tan \phi}, & u < 0 \\ \frac{1 - U \tan \phi}{u \tan \beta}, & u > 0 \end{cases}$$

$$W'' = u'' \tan \beta \quad \text{and} \quad (U'') \tan \phi$$
Result a 500 for bluntness.

\[
\frac{W_{max}}{\frac{d}{2}} \text{ strongly increased by sweep and favorable of } \frac{dx}{\partial x}
\]

\(\Rightarrow\) steamline instability is suppressed in a strong favorable pressure gradient.

For a swept wing:

- fuller streamwise profile
- strong crossflow
- \(\Delta u(y)\)
- larger shape factor
- \(\Delta u(y)\) weaker crossflow

Drog polar of swept vs. unswept wings:

- \(C_p\) increase due to transition