Lecture 8

Thin Shear Layer Approximation

A) TSL Equations: Summary, Edge Conditions, Coordinate, Streamfunction
B) Shear Layer Categories and Boundary Conditions
C) Asymmetric form

Reading:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

unsteady \( \frac{\partial p}{\partial y} = 0 \)

Pressure at the edge of the TSL:

\[
\frac{\partial p}{\partial y} = 0
\]

\( \Rightarrow p(x, y) = p_e(x) \)

where \( p_e \) denotes edge of shear layer

\[
\frac{\partial p_e}{\partial x} = \frac{d p_e}{d x}
\]
Recall Bernoulli equation for steady incompressible incompressible flow can be applied along a streamline
\[ p_c(x) + \frac{1}{2} p_c (u_e^2 + v_e^2) = p_0 \]

\[ \Rightarrow \frac{dp_c}{dx} = -p_c \frac{dv_c}{dx} - p_c \frac{du_c}{dx} \]

Since \( v_e < u_e \)

\[ \frac{dp_c(x)}{dx} = -p_c \frac{dv_c}{dx} \]

(can also be obtained by considering x-mom)

(Compressible (for incompressible for Euler flow))

Now, given edge conditions, we can solve TSL equations to obtain layer behavior

Example \( u_e \)

(Reasonable approx. \( u \)

(Compressible flow)

TSL coordinates are not Cartesian. They are typically streamline and normal coordinates \((s, n)\), oriented so that \( n \ll n \)

Note: Incomp vs compressible form of x-mom

\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u \frac{dv}{dx} + \frac{1}{p} \frac{\partial T}{\partial y} \text{ where } T = \rho \frac{\partial u}{\partial y} \]

\[ \frac{\partial u}{\partial y} + p \frac{\partial v}{\partial y} = p \frac{dv}{dx} + \frac{\partial T}{\partial y} \]
A stream function is a scalar function \( \psi(x,y) \) defined by:

\[
u = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}
\]

Substituting in the momentum eqm:

\[
\frac{\partial^2 \psi}{\partial y^2} \frac{\partial u}{\partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y} \left( \nu \frac{\partial u}{\partial y} \right)
\]

\[
\psi_y \cdot \psi_{xy} - \psi_x \psi_{yy} = \nu \frac{\partial u}{\partial x} + v \psi_y \psi_{yy}
\]

Note:

1. Continuity is no longer required.
2. Compressible stream function: \( \rho u = \psi_y, \rho v = -\psi_x \)

B) Stream layer, boundary conditions, and B.C.s.

TSL equations apply to very wide variety of flows. Different flows distinguished by boundary conditions.

3rd-order system requiring 3 B.C.s per X location.

1) Wall B.C.: \( @ y = 0 \) : \( u = 0, v = 0 \)
\( y = y_0 \) : \( u = \psi_y \)

2) Porous wall: \( @ y = 0 \) : \( u = 0, v = \nu w \)
\( y = y_0 \) : \( u = \psi_y \)
3) Wake or Jet:
(Symmetric)

@ y = 0: \( \frac{\partial u}{\partial y} = 0 \), \( v = 0 \)

@ y = y_0: \( u = uc \)

\[ \text{--- Wake} \]

\[ v = 0 \text{ for jet} \]

4) Wake:
(Geomral)

@ y = y_c: \( u = uc^+ \)

@ y = y_e: \( u = ye \)

@ y = y_i: \( v = ve \)

\( u \) is some arbitrary interior point, and \( v_e = v(y_e) \) is also arbitrary, as long as \( v_e < u_e \). Changing \( v_e \) merely reporions shear layer ivi \( x, y \) coordinates system.

Example:
\( y = 0, v = 0 \)

\[ \text{---} \]

(x, y) coordinates is ok as long a \( x \) is closely aligned with TSL, so that \( \frac{\partial}{\partial y} \gg \frac{\partial}{\partial x} \)

Assumption is valid.
Mixing Layer

\[ \begin{align*}
\gamma = \gamma^+ & : u = u^+ \\
\gamma = \gamma^0 & : v = v^0 \\
\gamma = \gamma^- & : u = u^-
\end{align*} \]

Axial symmetry TSL's

\[ \begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \\
\frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} &= \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial w}{\partial y} \right) \\
\frac{\partial p}{\partial y} &= 0 \\
v \cdot \hat{u} &= 0 \\
p = \varphi(x, z) \text{ and } \hat{u} = u \hat{i} + w \hat{k}
\end{align*} \]

\( \hat{z} \) - spanwise direction or a swept wing (away from root)

Special case of steady shear layer - axial symmetric - gradient around the circumference are zero (flow in a duct, wing-body junction)

\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho \gamma} \frac{\partial}{\partial y} \left( \gamma \mu \frac{\partial u}{\partial y} \right) \]

\[ \frac{\partial}{\partial x} (r^2 u) + \frac{\partial}{\partial y} (r v) = 0 \]

Note \( \varphi(x) \approx \delta / \delta_0 \)

need not be small

local streamwise curvature may result in \( \frac{\partial}{\partial y} \neq 0 \)
\[ r = r_0 + y \cos \theta \quad \text{tan} \theta = \frac{dr}{dz} \]

Define \( t = \frac{y \cos \theta}{r_0} \Rightarrow \frac{r}{r_0} = 1 + t \)

\[ \text{Consider an alternative form} \]

If \( t \ll 1 \quad 8/\lambda_0 \ll 1 \)

Equation becomes

\[ \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) = 0 \]

\[ \frac{u^2}{\partial x} + \frac{v^2}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial}{\partial y} \left( \rho \frac{\partial u}{\partial y} \right) \]

\[ + \frac{ue}{\partial x} + \frac{ve}{\partial y} \]

Does not apply as axisymmetric jet \( r \rightarrow 0 \). In that case,

\[ \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0 \]

\[ \frac{\partial u}{\partial x} + \frac{v}{\partial r} \left( \frac{\partial u}{\partial r} \right) + \frac{1}{\rho \partial r} \left( \rho \frac{\partial u}{\partial r} \right) \]

\[ + \frac{ue}{\partial x} \]

Ax. For \( B \cdot C \quad r=0, \quad u=0, \quad \frac{\partial u}{\partial r} = 0, \quad r = R, \quad u = 0 \)
**Thin Shear Layer Equations and Boundary Conditions**

**TSL Approximations:**
\[
\frac{\partial^2 u}{\partial y^2} \gg \frac{\partial u}{\partial x^2}, \quad \text{transverse diffusion} \gg \text{streamwise diffusion}
\]
\[
\frac{\partial p}{\partial x} = \text{constant in } y, \quad \text{so } p(x,y) = p(x), \quad \frac{\partial p}{\partial x} = \frac{dp}{dx} = -\rho u \frac{du}{dx}
\]

**TSL Equations:**
- **Continuity:** \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
- **x-Momentum:** \[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + v \frac{\partial^2 u}{\partial y^2} \]

**Boundary Conditions:**
- 3rd-order system, needs 3 BCs per x location

**Wall Boundary Layer**
1. \( y = y_e: \quad u = u_e \) or \( v = v_e \)
2. \( y = 0: \quad u = 0 \)
3. \( y = 0: \quad v = 0 \)

**Wake**
1. \( y = y_e+ : \quad u = u_e \)
2. \( y = y_0: \quad v = v_0 \)
3. \( y = y_e- : \quad u = u_e \)

**Jet**
1. \( y = y_e+: \quad u = 0 \)
2. \( y = y_0: \quad v = v_0 \)
3. \( y = y_e- : \quad u = 0 \)

**Mixing Layer**
1. \( y = y_e+: \quad u = u_e+ \)
2. \( y = y_0 : \quad v = v_0 \)
3. \( y = y_e- : \quad u = u_e- \)

**Porous Wall B.L.**
1. \( y = y_e: \quad u = u_e \)
2. \( y = 0: \quad u = 0 \)
3. \( y = 0: \quad v = v_w \)