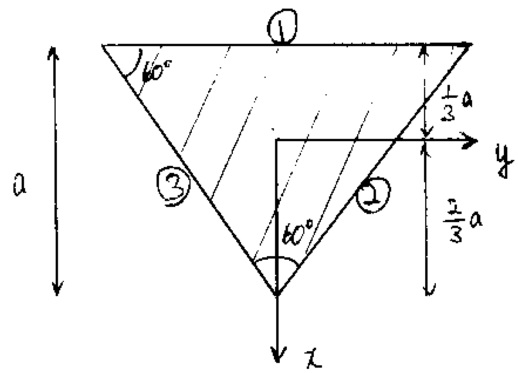
Solutions to Home Assignment #6

Warm-up Exercises



Stress function :

$$\phi(x,y) = C_1\,(x - \sqrt{3}\,y - 2/3\,a)(x + \sqrt{3}\,y - 2/3\,a)(x + 1/3\,a)$$

1. In order for the given stress function to be valid, $\phi(x,y)$ has to be zero on the surface of the rod, i.e.,

$$\phi(x,y) = 0 \quad \text{on surface} \quad \text{———} \; ①$$

Since there are three sides to the triangle, we will consider each side seporately.

<u>Surface ①</u> : On this surface, $x = -\frac{1}{3}a$. Plugging this into the stress function, we get,

$$\phi(x=-\tfrac{1}{3}a,\ y) = 0$$

So, equation ① is satisfied.

Surface ② : This surface passes through the two points.

$$(x, y) = \left(\tfrac{2}{3}a, 0\right)$$
$$(x, y) = \left(0, \tfrac{2}{3\sqrt{3}}a\right)$$

The equation for a line passing through the two points is

$$y = -\frac{1}{\sqrt{3}}x + \frac{2}{3\sqrt{3}}a \qquad \text{——— ②}$$

Plugging equation ② into the stress function we get,

$$\phi = C_1\left(x-\sqrt{3}\left(-\tfrac{1}{\sqrt{3}}x+\tfrac{2}{3\sqrt{3}}a\right)-\tfrac{2}{3}a\right)\left(x+\sqrt{3}\left(-\tfrac{1}{\sqrt{3}}x+\tfrac{2}{3\sqrt{3}}a\right)-\tfrac{2}{3}\right)\left(x+\tfrac{1}{3}a\right)$$
$$= C_1\left(x+\left(x-\tfrac{2}{3}a\right)-\tfrac{2}{3}a\right)\underbrace{\left(\cancel{x}+\left(-\cancel{x}+\cancel{\tfrac{2}{3}a}\right)-\cancel{\tfrac{2}{3}a}\right)}_{=0}\left(x+\tfrac{1}{3}a\right)$$

$$\therefore\ \phi = 0$$

So, equation ① is saticfied.

Surface ③ : This surface passes through the two points,

$$(x, y) = \left(\tfrac{2}{3}a, 0\right)$$
$$(x, y) = \left(0, -\tfrac{2}{3\sqrt{3}}a\right)$$

The equation for a line passing through the two points is

$$y = \frac{1}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}a \quad \text{———————} \quad ③$$

Plugging equation ③ into the stress function, we get,

$$\phi = C_1\left(x - \sqrt{3}\left(\frac{1}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}a\right) - \frac{2}{3}a\right)\left(x + \sqrt{3}\left(\frac{1}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}a\right) - \frac{2}{3}a\right)\left(x + \frac{1}{3}a\right)$$

$$= C_1\underbrace{\left(\cancel{x} - \left(\cancel{x} - \cancel{\tfrac{2}{3}a}\right) - \cancel{\tfrac{2}{3}a}\right)}_{=0}\left(x + \left(x - \frac{2}{3}a\right) - \frac{2}{3}a\right)\left(x + \frac{1}{3}a\right)$$

$$\therefore \quad \phi = 0$$

So, equation ① is satisfied.

2. To find the torsional constant, J, we will use the relation

$$J = \frac{T}{GK} \quad \text{———————} \quad ①$$

where T is the applied torque, G is the shear modulus and K is the rate of twist. So, we need to find GK and T.

From Poisson's equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2GK$$

$$\Rightarrow GK = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \quad \text{———————} \quad ②$$

The torque, T, is equal to

$$T = -2\iint \phi \, dx\, dy \quad \text{———} \quad ③$$

Substituting equations ② and ③ into equation ①, we get,

$$J = \frac{-2\iint \phi \, dx\, dy}{\frac{1}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)} \quad \text{———} \quad ④$$

So, we need to evaluate equations ② and ③ to find J.

In order to do this, we'll first expand the stress function, ϕ, in polynomial form, i.e.,

$$\phi(x,y) = C_1\left(x - \sqrt{3}y - \frac{2}{3}a\right)\left(x + \sqrt{3}y - \frac{2}{3}a\right)\left(x + \frac{1}{3}a\right)$$

$$= C_1\left(\frac{4a^3}{27} - ax^2 + x^3 - ay^2 - 3xy^2\right) \quad \text{———} \quad ⑤$$

Now, let's evaluate GK using equation ②.

$$\frac{\partial^2\phi}{\partial x^2} = C_1(-2a + 6x) \quad , \quad \frac{\partial^2\phi}{\partial y^2} = C_1(-2a - 6x)$$

$$\Rightarrow \quad GK = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right) = -2aC_1 \quad \text{———} \quad ⑥$$

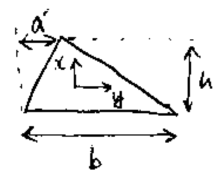
Next, let's evaluate T using equation ③.

$$\iint \phi \, dx\,dy = C_1 \iint \left(\frac{4a^3}{27} - ax^2 + x^3 - ay^2 - 3xy^2 \right) dx\,dy$$

$$= \frac{4a^3}{27} C_1 \iint dx\,dy - aC_1 \iint x^2 dx\,dy$$

$$+ C_1 \iint x^3 dx\,dy - aC_1 \iint y^2 dx\,dy - 3C_1 \iint xy^2 dx\,dy \quad \text{—①}$$

Some of the double integrals in equation ① have physical meanings :



$$I_y = \iint x^2 dx\,dy = \frac{1}{36} bh^3 \quad \text{for a triangle}$$

area moments of inertia

$$I_x = \iint y^2 dx\,dy = \frac{1}{36}(b^3h - b^2ha' + bha'^2) \quad \text{for a triangle}$$

$$\text{Area} = \iint dx\,dy = \frac{1}{2} bh \quad \text{for a triangle.}$$

Using the above definitions,

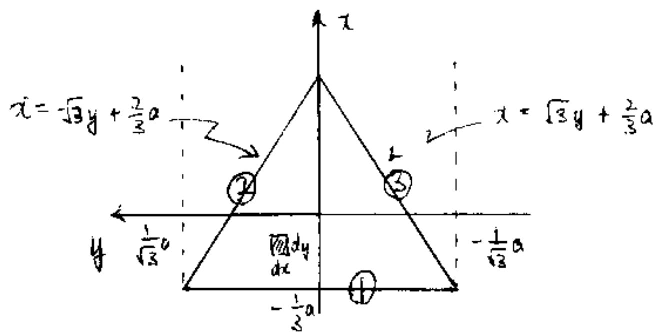
$$* \begin{cases} b = \frac{2}{\sqrt{3}} a \\ h = a \\ a' = \frac{1}{\sqrt{3}} a \end{cases}$$

$$\iint x^2 dx\,dy = \frac{1}{36}\left(\frac{2}{\sqrt{3}}a\right)(a)^3 = \frac{1}{18\sqrt{3}} a^4$$

$$\iint y^2 dx\,dy = \frac{1}{36}\left[\left(\frac{2}{\sqrt{3}}\right)^3 - \left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)^2\right] = \frac{1}{18\sqrt{3}} a^4$$

$$\iint dx\,dy = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{\sqrt{3}}a\right)a = \frac{1}{\sqrt{3}} a^2$$

The two remaining integrals need to be evaluated by integration. Noting the integration limits from the diagram below (geometry rotated 180° for convenience), we get,



$$\iint x^3 \, dx \, dy = \int_{-\frac{1}{\sqrt{3}}a}^{0} \int_{-\frac{1}{3}a}^{\sqrt{3}y + \frac{2}{3}a} x^3 \, dx \, dy + \int_{0}^{\frac{1}{\sqrt{3}}a} \int_{-\frac{1}{3}a}^{-\sqrt{3}y + \frac{2}{3}a} x^3 \, dx \, dy$$

$$= \int_{-\frac{1}{\sqrt{3}}a}^{0} \frac{1}{4} x^4 \Big|_{-\frac{1}{3}a}^{\sqrt{3}y + \frac{2}{3}a} dy + \int_{0}^{\frac{1}{\sqrt{3}}a} \frac{1}{4} x^4 \Big|_{-\frac{1}{3}a}^{-\sqrt{3}y + \frac{2}{3}a} dy$$

$$= \int_{-\frac{1}{\sqrt{3}}a}^{0} \frac{1}{4}\left[\left(\sqrt{3}y + \frac{2}{3}a\right)^4 - \left(\frac{1}{3}a\right)^4\right] dy + \int_{0}^{\frac{1}{\sqrt{3}}a} \frac{1}{4}\left[\left(-\sqrt{3}y + \frac{2}{3}a\right)^4 - \left(\frac{1}{3}a\right)^4\right] dy$$

$$= \frac{1}{4}\left[\frac{1}{5\sqrt{3}}\left(\sqrt{3}y + \frac{2}{3}a\right)^5 - \left(\frac{1}{3}a\right)^4 y\right]_{-\frac{1}{\sqrt{3}}a}^{0}$$

$$+ \frac{1}{4}\left[-\frac{1}{5\sqrt{3}}\left(-\sqrt{3}y + \frac{2}{3}a\right)^5 - \left(\frac{1}{3}a\right)^4 y\right]_{0}^{\frac{1}{\sqrt{3}}a}$$

$$= \frac{1}{4}\left[\frac{1}{5\sqrt{3}}\left(\frac{2}{3}a\right)^5 - \frac{1}{5\sqrt{3}}\left(-a + \frac{2}{3}a\right)^5 - \left(\frac{1}{3}a\right)^4 \frac{1}{\sqrt{3}}a\right]$$

$$+ \frac{1}{4}\left[-\frac{1}{5\sqrt{3}}\left(-a + \frac{2}{3}a\right)^5 - \left(\frac{1}{3}a\right)^4 \frac{1}{\sqrt{3}}a + \frac{1}{5\sqrt{3}}\left(\frac{2}{3}a\right)^5\right]$$

$$\Rightarrow \quad \iint x^3\,dx\,dy = \frac{a^5}{135\sqrt{3}}$$

Similarly,

$$\iint x y^2\,dx\,dy = \int_{-\frac{1}{\sqrt{3}}a}^{0}\int_{-\frac{1}{3}a}^{\sqrt{3}y+\frac{2}{3}a} x y^2\,dx\,dy + \int_{0}^{\frac{1}{\sqrt{3}}a}\int_{-\frac{1}{3}a}^{-\sqrt{3}y+\frac{2}{3}a} x y^2\,dx\,dy$$

$$\Rightarrow \quad \iint x y^2\,dx\,dy = -\frac{a^5}{135\sqrt{3}}$$

Therefore, plugging the integrals into equation ①, we get,

$$\iint \phi\,dx\,dy = C_1\left[\frac{4}{27}a^3\left(\frac{1}{\sqrt{3}}a^2\right) - a\left(\frac{1}{18\sqrt{3}}a^4\right) + \frac{a^5}{135\sqrt{3}} - a\left(\frac{1}{18\sqrt{3}}a^4\right) + 3\left(\frac{a^5}{135\sqrt{3}}\right)\right]$$

$$= a^5 C_1\left[\frac{4}{27\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{4}{135\sqrt{3}}\right]$$

$$\therefore \quad \iint \phi\,dx\,dy = \frac{a^5 C_1}{15\sqrt{3}}$$

Thus, from equation ③,

$$T = -2\left(\frac{a^5 C_1}{15\sqrt{3}}\right) \qquad \text{———} \quad ⑧$$

Using equation ④ (or ①), we finally get,

$$J = \frac{-\cancel{2}\left(\frac{a^{\cancel{5}}\cancel{C}_1}{15\sqrt{3}}\right)}{-\cancel{2}\,\cancel{a}\,\cancel{C}_1}$$

$$\boxed{J = \frac{a^4}{15\sqrt{3}}}$$

3. The rate of twist can be found by using the results of problem 2. Equation ① from problem 2 says

$$J = \frac{T}{GK}$$

$$\Rightarrow \quad K = \frac{T}{GJ} \qquad , \; K = \frac{d\alpha}{dz} = \text{rate of twist} \quad \text{①}$$

Therefore, the rate of twist is

$$\frac{d\alpha}{dz} = \frac{T}{GJ} = \frac{T}{G\left(\frac{a^4}{15\sqrt{3}}\right)} = \frac{15\sqrt{3}\,T}{Ga^4}$$

$$\Rightarrow \quad \boxed{\frac{d\alpha}{dz} = \frac{15\sqrt{3}\,T}{Ga^4}}$$

4. In order to find the iso-resultant stress contours and the magnitude of the maximum resultant stress, we need to find the stress components, σ_{xz} and σ_{yz}. Note that the

constant, C_1, in the stress function can be obtained from equation ⑥ of problem 2.

$$GK = -2aC_1$$

$$\Rightarrow \quad C_1 = -\frac{1}{2a}GK \qquad \text{①}$$

Plugging equation ① into the stress function, ϕ, we get

$$\phi(x,y) = -\frac{1}{2a}GK\left(\frac{4a^3}{27} - ax^2 + x^3 - ay^2 - 3xy^2\right)$$

The shear stresses are

$$\sigma_{xz} = -\frac{\partial\phi}{\partial y} = \frac{1}{2a}GK\left(-2ay - 6xy\right) = \frac{GK}{a}\left(-ay - 3xy\right) \qquad \text{②}$$

$$\sigma_{yz} = \frac{\partial\phi}{\partial x} = \frac{1}{2a}GK\left(-2ax + 3x^2 - 3y^2\right) \qquad \text{③}$$

The resultant stress is

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{1}{2a}GK\left[\left(2ay + 6xy\right)^2 + \left(-2ax + 3x^2 - 3y^2\right)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{2a}GK\left[4a^2(x^2+y^2) - 12a(x^3 - 3xy^2) + 9(x^2+y^2)^2\right]^{\frac{1}{2}} \qquad \text{④}$$

The magnitude of the maximum resultant stress can be obtained by considering the expression in brackets (under the square root sign). It can be inferred from the general form of this expression that the maximum resultant lies somewhere on the boundaries of the triangle because the first term → $4a^2(x^2+y^2)$, and the last term → $9(x^2+y^2)^2$, increases away from the mid-point). This also agrees with the intuition that away from the center of the triangle, the stresses will be greater when the bar is under torsion. So, lets plug in the upper boundary, $x = -\frac{1}{3}a$, into equation ④ and maximizing with respect to y.

$$f(x=-\tfrac{1}{3}a, y) = 4a^2\left(\tfrac{1}{9}a^2+y^2\right) - 12a\left(-\tfrac{1}{27}a^3 + ay^2\right) + 9\left(\tfrac{1}{9}a^2+y^2\right)^2$$

$$\Rightarrow f(y) = \frac{4}{9}a^4 + 4a^2y^2 + \frac{\cancel{12}^{4}}{\cancel{27}_{9}}a^4 - 12a^2y^2 + \frac{1}{9}a^4 + 2a^2y^2 + 9y^4$$

$$= a^4 - 6a^2y^2 + 9y^4 \qquad \text{------} \quad ⑤$$

$$\Rightarrow \quad f'(y) = -12a^2 y + 36y^3 = 0$$

$$\Rightarrow \quad \left(\begin{array}{l} y = 0 \\ y = \pm \frac{a}{\sqrt{3}} \end{array} \right) \text{ minimum or maximum points.}$$

Plugging these values back into equation ⑤, we get

$$f(0) = a^4 \quad \longleftarrow \text{maximum}$$

$$f\left(\pm\frac{a}{\sqrt{3}}\right) = a^4 - 6a^2\left(\frac{a^2}{3}\right) + 9\left(\frac{a^4}{9}\right) = 0 \quad \leftarrow \text{minimum.}$$

Therefore, the maximum resultant stress occurs at

$$\left(\begin{array}{l} x = -\frac{1}{3}a \\ y = 0 \end{array} \right.$$

and the magnitude is

$$\tau_{max} = \frac{1}{2a} GK \left[f(0) \right]^{\frac{1}{2}} = \frac{1}{2a} GK a^2$$

$$\therefore \quad \boxed{\tau_{max} = \frac{a}{2} GK}$$

* Note: the location of maximum resultant stress occurs at two other points due to symmetry. The mid-points of each side of the triangle are all location of maximum resultant stress. This can be shown mathematically by plugging in the equations of each side (equations ② and ③

from problem 1), and then maximizing with respect to y.

The iso-resultant stress contours can be generated using equation ④. The figure below was plotted for $a=1$ and $GK=1$.