Lecture 13  Gauss' Method for the Time-Constrained BVP

# 7.3 Lagrange’s equations for the boundary-value problem

\begin{align*}
\sqrt{\mu} (t_2 - t_1) &= 2a^2 (\psi - \sin \psi \cos \phi) \quad (1) \\
r_1 + r_2 &= 2a(1 - \cos \psi \cos \phi) \quad (2) \\
\sqrt{r_1 r_2} \cos \frac{1}{2} \theta &= a(\cos \psi - \cos \phi) \quad (3)
\end{align*}

Gauss’s Equation for the Semimajor Axis

Eliminate \( \cos \phi \) between (2) and (3):

\[ \frac{1}{a} = \frac{2 \sin^2 \psi}{r_1 + r_2 - 2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta \cos \psi} \]

When \( \psi \) and \( \theta \) are very small (of the order of 2 or 3 degrees), then \( r_1 \) and \( r_2 \) will have almost the same value. The denominator will be determined as the difference between two almost equal terms resulting in a severe loss of accuracy. To prevent this, Gauss wrote

\[ \frac{1}{a} = \frac{\sin^2 \psi}{2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta (\ell + \sin^2 \frac{1}{2} \psi)} \quad (4) \]

where \( \ell \) is defined as

\[ \ell = \sqrt{\frac{r_2}{r_1} + \frac{r_1}{r_2}} - \frac{1}{2} \]

The problem of subtracting two almost equal quantities still exists but Gauss had a different method for calculating \( \ell \) which avoided any subtraction:

\[ \ell = \frac{\sin^2 \frac{1}{2} \theta + \tan^2 2\omega}{\cos \frac{1}{2} \theta} \quad \text{where} \quad \tan(\frac{1}{4} \pi + \omega) = \left(\frac{r_2}{r_1}\right)^{\frac{1}{4}} \]

An alternate method, which does not require any inverse trigonometric function is to express \( r_2 = r_1 (1 + \epsilon) \)

The quantity \( \epsilon \) is simply the fractional part resulting when \( r_2 \) is divided by \( r_1 \) (assuming, of course, that \( r_2 \) exceeds \( r_1 \)). The result is

\[ \tan^2 2\omega = \frac{\frac{1}{4} \epsilon^2}{\sqrt{\frac{r_2}{r_1} + \frac{r_2}{r_1} \left(2 + \sqrt{\frac{r_2^2}{r_1^2}}\right)}} \]
Gauss’s Time Equation

Eliminate $\cos \phi$ between (1) and (3):

$$\sqrt{\frac{\mu}{a^3}} (t_2 - t_1) = 2\psi - \sin 2\psi + \frac{2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta}{a} \sin \psi$$  \hfill (5)

Next using Eq. (4) for $1/a$ he obtained

$$\frac{\sqrt{\mu(t_2 - t_1)} \sin^3 \psi}{(2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta)^2 (\ell + \sin^2 \frac{1}{2} \psi)^2} = \frac{m \sin^3 \psi}{(\ell + \sin^2 \frac{1}{2} \psi)^2} = 2\psi - \sin 2\psi + \frac{\sin^3 \psi}{\ell + \sin^2 \frac{1}{2} \psi}$$

where he defined

$$m = \frac{\sqrt{\mu(t_2 - t_1)}}{(2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta)^2}$$

which requires that $0 < \theta < 180^\circ$.

Finally, Gauss defined

$$y^2 = \frac{m^2}{\ell + \sin^2 \frac{1}{2} \psi}$$

so that the time equation (5) can be written as

$$m \times \frac{y^3}{m^3} = 2\psi - \sin 2\psi + \frac{y^2}{m^2} \quad \text{or} \quad y^3 - y^2 = m^2 \frac{2\psi - \sin 2\psi}{\sin^3 \psi}$$

which are Gauss’ equations, to be solved for $y$ and $\psi$.

**The Orbital Parameter and the Significance of $y$**

From Lecture 9 on Page 3

$$p = \frac{\sin \phi}{\sin \psi} p_m = \frac{\sin \phi}{\sin \psi} \times \frac{2r_1 r_2 \sin^2 \frac{1}{2} \theta}{c} = \frac{\sin \phi}{\sin \psi} \times \frac{2r_1 r_2 \sin^2 \frac{1}{2} \theta}{2a \sin \psi \sin \phi} = \frac{r_1 r_2 \sin^2 \frac{1}{2} \theta}{a \sin^2 \psi}$$

Then from

$$a = \frac{2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta (\ell + \sin^2 \frac{1}{2} \psi)}{\sin^2 \psi}$$

we have

$$a \sin^2 \psi = \frac{2m^2 \sqrt{r_1 r_2} \cos \frac{1}{2} \theta}{y^2} \quad \text{where} \quad m^2 = \frac{\mu(t_2 - t_1)^2}{(2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta)^3}$$

so that

$$p = \frac{r_1^2 r_2^2}{\mu(t_2 - t_1)^2} \frac{y^2 \sin^2 \theta}{\mu} = \frac{h^2}{\mu}$$

from which

$$\frac{1}{2} \frac{h(t_2 - t_1)}{r_1 r_2 \sin \theta} = \frac{\text{Area of sector}}{\text{Area of triangle}}$$

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Changing the independent variable from $\psi$ to $x = \sin^2 \frac{1}{2} \psi$

Define

$$Q = \frac{2\psi - \sin 2\psi}{\sin^3 \psi}$$

then

$$3Q \sin^2 \psi \cos \psi + \sin^3 \psi \frac{dQ}{d\psi} = 2 - 2 \cos 2\psi = 4 \sin^2 \psi$$

Now

$$\frac{dx}{d\psi} = \sin \frac{1}{2} \psi \cos \frac{1}{2} \psi = \frac{1}{2} \sin \psi$$

so that

$$3Q \cos \psi + \frac{1}{2} \sin^2 \psi \frac{dQ}{dx} = 4$$

Since

$$\cos \psi = 1 - 2 \sin^2 \frac{1}{2} \psi = 1 - 2x$$

$$\sin^2 \psi = 4 \sin^2 \frac{1}{2} \psi (1 - \sin^2 \frac{1}{2} \psi) = 4x(1 - x)$$

then

$$2x(1 - x) \frac{dQ}{dx} = 4 - (3 - 6x)Q$$

Write

$$Q = \frac{4}{3}(1 + q_1 x + q_2 x^2 + q_3 x^3 + q_4 x^4 + \cdots)$$

Substitute and equate like powers of $x$ to obtain:

$$q_1 = \frac{6}{5} \quad q_2 = \frac{8}{7} q_1 \quad q_3 = \frac{10}{9} q_2 \quad q_4 = \frac{12}{11} q_3 \quad \text{etc.}$$

resulting in

$$Q(x) = \frac{4}{3} F(3, 1; \frac{5}{2}; x) = \frac{4}{3} \left( 1 + \frac{6}{5} x + \frac{6 \cdot 8}{5 \cdot 7} x^2 + \frac{6 \cdot 8 \cdot 10}{5 \cdot 7 \cdot 9} x^3 + \frac{6 \cdot 8 \cdot 10 \cdot 12}{5 \cdot 7 \cdot 9 \cdot 11} x^4 + \cdots \right)$$

$$F(3, 1; \frac{5}{2}; x) = \frac{1}{1 - \frac{\gamma_1 x}{1 - \frac{\gamma_2 x}{1 - \frac{\gamma_3 x}{1 - \cdots}}}}$$

$$\gamma_n = \begin{cases} 
\frac{(n + 2)(n + 5)}{(2n + 1)(2n + 3)} & n \text{ odd} \\
\frac{n(n - 3)}{(2n + 1)(2n + 3)} & n \text{ even}
\end{cases}$$

a. The series converges for $-1 < x < 1$.

b. The continued fraction converges for $-\infty < x < 1$.

Note: The function $F(\alpha, \beta; \gamma; x)$ is Gauss’ Hypergeometric Function which we will exam in some detail in the next Lecture.
The Universal Form of Gauss’ Method

We can extend the definition of $x$ so that

$$x = \begin{cases} \sin^2 \frac{1}{4}(E_2 - E_1) & \text{ellipse} \\ 0 & \text{parabola} \\ -\sinh^2 \frac{1}{4}(H_2 - H_1) & \text{hyperbola} \end{cases}$$

The range of $x$ is $-\infty < x < 1$. The series representation of $Q(x)$ will not converge when $x < -1$. However the continued fraction does converge over the full range.

Possible Algorithm

Gauss’ equations are:

$$y^2 = \frac{m^2}{\ell + x} \quad \text{and} \quad y^3 - y^2 = m^2Q(x)$$

in terms of $x$. The following is a recursive algorithm for the solution:

1. Set $x = 0$
2. Solve of cubic $y^3 - y^2 = m^2Q(x)$

Note: Solution of the cubic: $y = 1 + \frac{4}{3} \sinh^2 \frac{1}{3}z$ where $\sinh z = \frac{3}{2} \sqrt{3m^2Q} $
3. Obtain new $x$ from

$$x = \frac{m^2}{y^2} - \ell$$

and repeat until the process converges.

Gauss’ Successive Substitution Algorithm \[0 < \theta < \pi\]

1. Given $r_1, r_2, \theta, \sqrt{\mu(t_2 - t_1)}$
2. Compute $\ell = \frac{r_1 + r_2}{4\sqrt{r_1r_2}\cos \frac{1}{2}\theta} - \frac{1}{2}$ and $m^2 = \frac{\mu(t_2 - t_1)^2}{(2\sqrt{r_1r_2}\cos \frac{1}{2}\theta)^3}$
3. Initialize $x = 0$
4. Calculate $\xi(x) = \frac{\frac{2}{35}x^2}{1 + \frac{2}{35}x - \frac{40}{63}x} - \frac{\frac{4}{99}x}{1 - \frac{70}{143}x}$

and $h = \frac{m^2}{6} + \ell + \xi(x)$
5. Solve the cubic $y^3 - y^2 - hy - \frac{h}{9} = 0$
6. Determine new $x = \frac{m^2}{y^2} - \ell$ and repeat until $x$ no longer changes.
7. Calculate the orbital elements:

$$\frac{1}{a} = \frac{8r_1r_2y^2x(1 - x)(1 + \cos \theta)}{\mu(t_2 - t_1)^2} \quad \text{and} \quad p = \frac{r_1^2r_2y^2\sin^2 \theta}{\mu(t_2 - t_1)^2}$$

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Avoiding the Continued Fraction When $\psi$ is Not Small

Instead of the continued fraction (which we shall learn more about in the next lecture), we can use the closed form expression

$$
\xi(\psi) = \frac{\sin^3 \psi - \frac{3}{4} (2\psi - \sin 2\psi)(1 - \frac{6}{5} \sin^2 \frac{1}{2} \psi)}{\frac{9}{10} (2\psi - \sin 2\psi)}
$$

Since $x = \sin^2 \frac{1}{2} \psi$, then $\psi = 2 \arcsin(\sqrt{x})$.

“The numerator of this expression is a quantity of the seventh order, the denominator of the third order, and $\xi$, therefore, of the fourth order, if $\psi$ is regarded as a quantity of the first order. Hence it is inferred that this formula is not suited to the exact numerical computation of $\xi$ when $\psi$ does not denote a very considerable angle.”

Karl Friedrich Gauss

Solving the Cubic Equation Pages 321 & 54

The solution of the cubic equation

$$
y^3 - y^2 - hy - \frac{1}{5} h = 0
$$

using the method developed on Page 321 of your textbook, is

$$
y = \frac{1}{3} (1 + w\sqrt{1 + 3h})
$$

where $w$ is the solution of

$$
w^3 - 3w = 2 \frac{1 + 6h}{(1 + 3h)^{\frac{3}{2}}} = 2b
$$

Note: Barker’s Equation is $w^3 + 3w = 2b$

We must address the cases $b < 1$ and $b \geq 1$ separately:

$b < 1$ Write $w = 2 \cos \frac{2}{3} x = 2(1 - 2 \sin^2 \frac{1}{3} x)$ and $b = \cos 2x = 1 - 2 \sin^2 x$

Then the cubic equation becomes $4 \cos^2 \frac{2}{3} x = 3 \cos \frac{2}{3} x = \cos 2x$ which is an identity for cosine functions. Hence:

$$
w = 2 \cos(\frac{1}{3} \arccos b)
$$

$b \geq 1$ Define $w = 2 \cosh \frac{2}{3} x = 2(1 + 2 \sinh^2 x)$ and $b = \cosh 2x = 1 + 2 \sinh^2 x$

Then the cubic equation becomes $4 \cosh^2 \frac{2}{3} x - 3 \cosh \frac{2}{3} x = \cosh 2x$ which is an identity for hyperbolic cosines. Hence:

$$
w = 2 \cosh(\frac{1}{3} \text{arccosh } b)
$$