Recall: Equation of orbit \( e \cdot r = p - r \) at \( P_1 \) and \( P_2 \) so that

\[
e \cdot r_1 = p - r_1 \\
e \cdot r_2 = p - r_2
\]

\[
\Rightarrow e \cdot (r_2 - r_1) = r_1 - r_2 \quad \text{or} \quad -e \cdot i = \frac{r_2 - r_1}{c}
\]

Orbital Elements of the Fundamental (Minimum Eccentricity) Ellipse

\[
e_F = \frac{|r_2 - r_1|}{c} \quad a_F = \frac{1}{2}(r_1 + r_2) \quad \frac{p_F}{p_m} = \frac{r_1 + r_2}{c}
\]

Since \( p_F = a_F(1 - e_F^2) \) and

\[
1 - e_F^2 = \frac{c^2 - (r_2 - r_1)^2}{c^2} = \frac{1}{c^2}(c + r_2 - r_1)(c + r_1 - r_2)
\]

\[
= \frac{4}{c^2}(s - r_1)(s - r_2) = \frac{4r_1r_2}{c^2} \sin^2 \frac{1}{2} \theta = \frac{2p_m}{c}
\]

From the figure at the top of this page, we see that \( \sin \phi_p = e_F \). Also the angle \( \omega_p \) is the complement of \( \phi_F \) so that \( \cos \omega_p = e_F \). Therefore:

The axes of the conjugate parabolic orbits coincide with the lines through the focus \( F \) and the extremities of the minor axis of the fundamental ellipse.
Locus of Mean Points  

Definition of mean point: At the point \( r_0 \), the velocity \( v_0 \) is parallel to the chord. Using the eccentricity vector at this point, we have \( v_0 \times h \cdot (r_2 - r_1) = 0 \) Therefore:

\[
\mu e \cdot (r_2 - r_1) = \left( v_0 \times h - \frac{\mu}{r_0} r_0 \right) \cdot (r_2 - r_1) = 0 - \frac{\mu}{r_0} r_0 \cdot (r_2 - r_1)
\]

Hence:

\[
\frac{e}{r_0} \cdot (r_2 - r_1) = \frac{1}{r_0} r_0 \cdot (r_2 - r_1) \quad \Rightarrow \quad r_0 \cdot (r_2 - r_1) = r_0 (r_2 - r_1)
\]

The loci of all mean points are the lines through the focus \( F \) and the extremities of the minor axis of the fundamental ellipse.

The Line Segment FS:

The line \( FS \) is the distance along mean point locus from the focus \( F \) to intersection with chord. The flight direction angle at \( P_0 \) is \( \gamma_0 \) and \( \delta \) is the angle opposite the line segment \( SP_1 \). Use the law of sines for the triangles:

\[
\Delta FP_1S: \quad \frac{FS}{\sin(\gamma_0 + \delta)} = \frac{r_1}{\sin \gamma_0} \quad \Delta FP_1P_2: \quad \frac{c}{\sin \theta} = \frac{r_2}{\sin(\gamma_0 + \delta)}
\]

and use the calculation on the previous page for \( 1 - e_F^2 \):

\[
\Delta FP_0C: \quad \sin \gamma_0 = \frac{b_F}{a_F} = \sqrt{1 - e_F^2} = \frac{2}{c} \sqrt{r_1 r_2 \sin \frac{1}{2} \theta}
\]

Therefore:

\[
FS = \sqrt{r_1 r_2} \cos \frac{1}{2} \theta
\]

Recall the proposition: Lecture 8, Page 2

The line connecting the focus and the point of intersection of the orbital tangents at the terminals bisects the transfer angle.

\[
\sqrt{r_1 r_2} = \begin{cases} FN \cos \frac{1}{2} (E_2 - E_1) & \text{ellipse} \\ FN & \text{parabola} \\ FN \cosh \frac{1}{2} (H_2 - H_1) & \text{hyperbola} \end{cases}
\]

Fig. 6.7 from An Introduction to the Mathematics and Methods of Astrodynamics. Courtesy of AIAA. Used with permission.
For the ellipse \( FN_1 \cos \frac{1}{2} (E_0 - E_1) = \sqrt{r_1 r_0} \) \( FN_2 \cos \frac{1}{2} (E_2 - E_0) = \sqrt{r_0 r_2} \)
and for the parabola \( FN_{1p} = \sqrt{r_1 r_{0p}} \) \( FN_{2p} = \sqrt{r_{0p} r_2} \)
Triangle \( \Delta FN_{1p}N_{2p} \) is similar to triangle \( \Delta FN_1N_2 \). Therefore
\[
\frac{FN_{1p}}{FN_{2p}} = \frac{FN_1}{FN_2} \implies \cos \frac{1}{2} (E_0 - E_1) = \cos \frac{1}{2} (E_2 - E_0)
\]

Hence
\[
E_0 = \frac{1}{2} (E_1 + E_2)
\]

The eccentric anomaly of the mean point of an orbit connecting two termini is the arithmetic mean between the eccentric anomalies of those termini.
Mean-Point Radius of the Parabolic Orbit  

Lecture 8, Page 1

The parameter of the parabola is obtained from

$\left( \frac{p_p}{p_m} \right)^2 - 2D \frac{p_p}{p_m} + 1 = 0$  
where  
$D = \frac{r_1 + r_2 - s(s - c)}{ac} = \frac{r_1 + r_2}{c}$

so that

$\frac{p_p}{p_m} = \frac{1}{c} \left( r_1 + r_2 + 2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta \right) = \frac{1}{c} \left( \sqrt{s + \sqrt{s - c}} \right)^2$

The mean point radius of the parabola is

$r_{0p} = \frac{p_p}{1 + \cos 2\phi_F} = \frac{p_p}{2 \cos^2 \phi_F} = \frac{p_p}{2(1 - e_F^2)} = \frac{p_p c}{4p_m}$

Hence

$$r_{0p} = \frac{1}{4} \left( r_1 + r_2 + 2\sqrt{r_1 r_2} \cos \frac{1}{2} \theta \right) = \frac{1}{2} (a_F + FS)$$

The mean point radius of the parabola extends to the midpoint between the chord and the extremity of the minor axis of the fundamental ellipse.

Mean-Point Radius of Ellipic and Hyperbolic Orbits  

Page 270

From the derivation of the eccentric anomaly of the mean point, we have

$FN_2 \cos \frac{1}{2}(E_2 - E_0) = FN_2 \cos \frac{1}{2} \left[ E_2 - \frac{1}{2} (E_1 + E_2) \right] = FN_2 \cos \frac{1}{4}(E_2 - E_1) = \sqrt{r_0 r_2}$

so that

$$\frac{FN_2}{FN_{2p}} \cos \frac{1}{4}(E_2 - E_1) = \sqrt{\frac{r_0}{r_{0p}}}$$

But, from similar triangles,

$$\frac{FN_2}{FN_{2p}} = \frac{r_0}{r_{0p}}$$

Therefore, we have the truly elegant expression

$r_0 = r_{0p} \sec^2 \frac{1}{4} (E_2 - E_1) = r_{0p} \sec^2 \frac{1}{2} \psi = r_{0p} (1 + \tan^2 \frac{1}{2} \psi)$

and, as we might expect,

$r_0 = r_{0p} \sech^2 \frac{1}{4} (H_2 - H_1)$

obtains also for hyperbolic orbits.

$$r_0 = \begin{cases} 
  a \left[ 1 - e \cos \frac{1}{2} (E_1 + E_2) \right] = a (1 - \cos \phi) & 
  \left\{ \begin{array}{l} 
  r_{0p} \left( 1 + \tan^2 \frac{1}{2} \psi \right) \\
  r_{0p} \left( 1 - \tanh^2 \frac{1}{4} (H_2 - H_1) \right) 
  \end{array} \right. 
\end{cases}$$