Formulating Hypothesis Testing Problems

Hypotheses about a random variable $x$ are often formulated in terms of its distributional properties. Example, if property is $a$:

Null hypothesis $H_0: a = a_0$

Objective of **hypothesis testing** is to decide whether or not to **reject** this hypothesis. Decision is based on estimator $\hat{a}$ of $a$:

Reject $H_0$: If observed estimate $\hat{a}$ lies in **rejection region** $R_{a0}$ ($\hat{a} \in R_{a0}$)

Do not reject $H_0$: Otherwise ($\hat{a} \notin R_{a0}$)

Select rejection region to obtain desired error properties:

| True situation | Test Result | $P(H_0|H_0) = 1 - \alpha$ | $P(\neg H_0|H_0) = \alpha$ (Type I Error) |
|----------------|-------------|---------------------------|------------------------------------------|
| $H_0$ true     | Do not reject $H_0$ | $\hat{a} \notin R_{a0}$ | Reject $H_0$ | $\hat{a} \in R_{a0}$ |
| $H_0$ false    | $P(H_0|\neg H_0) = \beta$ (Type II Error) | $P(\neg H_0|\neg H_0) = 1 - \beta$ |

Type I error probability $\alpha$ is called the test **significance level**.

Deriving Hypothesis Rejection Regions for Large Sample Tests

Hypothesis test is often based on a **standardized statistic** that depends on unknown true property and its estimate. Basic concepts are the same as used to derive confidence intervals (see Class 14).

An example is the $z$ statistic:

$$z(\hat{a}, a) = \frac{\hat{a} - a}{SD(\hat{a})}$$

If the estimate is unbiased $E[z] = 0$ and $Var[z] = 1$. 

Define a rejection region $R_{z_0}$ in terms of $z$ as:

$$R_{z_0} : \begin{align*}
    z(\hat{\alpha}, a_0) &\leq z_L \\
    z(\hat{\alpha}, a_0) &\geq z_U
\end{align*}$$

As rejection region grows Type I error increases and Type II error decreases (test is more likely to reject hypothesis).

As rejection region shrinks Type I error decreases and Type II error increases (test is less likely to reject hypothesis).

Usual practice is to select rejection region to insure that Type I error probability is equal to a specified value $\alpha$.

For a two-sided test require that Type I error probability is distributed equally between intervals below $z_L$ (probability $= \alpha/2$) and above $z_U$ (probability $= \alpha/2$).

These probabilities are:

$$P[z(\hat{\alpha}, a_0) \leq z_L | H0] = P[z(\hat{\alpha}, a_0) \leq z_L] = F_z(z_L) = \frac{\alpha}{2}$$

$$P[z(\hat{\alpha}, a_0) \geq z_U | H0] = P[z(\hat{\alpha}, a_0) \geq z_U] = 1 - F_z(z_U) = \frac{\alpha}{2}$$

$$z_L = F_z^{-1}\left(\frac{\alpha}{2}\right) \quad z_U = F_z^{-1}\left(1 - \frac{\alpha}{2}\right)$$

For large samples $z(\hat{\alpha}, a_0)$ has a unit normal distribution. Use the MATLAB function `norminv` to evaluate $F_z^{-1}$.

If the definition of $z$ is applied a two-sided rejection region $R_{a0}$ can also be written directly in terms of the estimate $\hat{\alpha}$:

$$R_{a0} : \begin{align*}
    \hat{\alpha} &\leq a_L = a_0 + F_z^{-1}\left(\frac{\alpha}{2}\right)SD[\hat{\alpha}] \\
    \hat{\alpha} &\geq a_U = a_0 + F_z^{-1}\left(1 - \frac{\alpha}{2}\right)SD[\hat{\alpha}]
\end{align*}$$

p Values

p value is largest significance level resulting in acceptance of $H0$.

For a symmetric two-sided rejection region and a large sample:
\[ p / 2 = 1 - F_z \left( \frac{\hat{a} - a_0}{SD(\hat{a})} \right) \quad \hat{a} \geq a \]
\[ p / 2 = F_z \left( \frac{\hat{a} - a_0}{SD(\hat{a})} \right) \quad \hat{a} \leq a_0 \]

For large samples use the MATLAB function `normcdf` to compute \( p \) from \( \hat{a} \) and \( SD[\hat{a}] \).

**Special Case -- Sample mean**

Consider hypothesis about value of population mean \( a = E[x] \):

\[ H_0: a = E[x] = a_0 \]

Base test on sample mean estimator \( m_x \). Obtain \( SD[m_x] \) from sample standard deviation:

\[ SD[m_x] = \frac{SD[x]}{\sqrt{N}} \approx \frac{s_x}{\sqrt{N}} \]

**Example: Testing whether mean is significantly different from zero**

Suppose \( a_0 = 0 \), \( s_x = 3 \), \( N = 9 \), \( m_x = 1.2 \) and \( \alpha = .05 \):

\[ R_{a_0}: m_x \leq a_L = 0 + F_z^{-1} \left( \frac{0.05}{2} \right) \frac{3}{\sqrt{9}} = -1.96 \]
\[ m_x \geq a_U = 0 + F_z^{-1} \left( 1 - \frac{0.05}{2} \right) \frac{3}{\sqrt{9}} = +1.96 \]

In this case hypothesis is **not rejected** since \( m_x = 1.2 \) does not lie in \( R_{a_0} \). The two-sided \( p \)-value is (see plot):

\[ 1 - p / 2 = F_z \left( \frac{m_x - a_0}{s_x / \sqrt{N}} \right) = F_z \left( 1.2 - 0 \right) \left[ \frac{3}{\sqrt{9}} \right] = F_z \left[ 1.2 \right] = .89 \]

\[ p = 0.22 \]
Normal Probability Plot

1 - p/2

Probability

Data