Problem statement

Isotropic solid (soil) on a rigid substrate (infinitely large in x-y-directions) $K, G$ given

Note: $p$ is applied pressure at the top of the soil layer

Goal: Determine $\xi(\vec{x}), \varepsilon(\vec{x}), \sigma(\vec{x})$

On the next few slides we will go through steps 1, 2, 3 and 4 to solve this problem.
Reminder: 4-step procedure to solve elasticity problems

- **Step 1**: Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!)

- **Step 2**: Write governing equations for stress tensor, strain tensor, and constitutive equations that link stress and strain, simplify expressions

- **Step 3**: Solve governing equations (e.g. by integration), typically results in expression with unknown integration constants

- **Step 4**: Apply BCs (determine integration constants)

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**Step 1**: Boundary conditions

Write out all BCs in mathematical equations

**Displacement BCs**: At z=H: Displacement specified

\[
\begin{align*}
\vec{\varepsilon}^d(z = H) &= (0,0,0) \quad \text{or} \quad \vec{\varepsilon}_x^d = 0, \vec{\varepsilon}_y^d = 0, \vec{\varepsilon}_z^d = 0
\end{align*}
\]

(no displacement at the interface between the soil layer and the rigid substrate)

**Stress BCs**: At z=0: Stress vector provided

\[
\vec{T}^d(\vec{n} = -\vec{e}_z, z = 0) = p\vec{e}_z
\]

Note: Orientation of surface and C.S.
Step 2: Governing equations

Write out all governing equations and simplify

Due to the symmetry of the problem (infinite in x- and y-directions), the solution will depend on z only, and there are no displacements in the x- and y-directions (anywhere in the solution domain): \( \xi = \xi_z \hat{z} \)

Governing eqn. for strain tensor:

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \xi_j}{\partial x_i} + \frac{\partial \xi_i}{\partial x_j} \right)
\]

Calculation of strain tensor simplifies (symmetry):

\[
\varepsilon_{zz} = \frac{\partial \xi_z}{\partial z} \quad (*)
\]

Note: only 1 nonzero coefficient of strain tensor

Governing eqn. for stress tensor:

\[
\text{div} \sigma + \rho g = 0
\]

(continues on next slide)

Step 2: Governing equations (cont’d)

Gravity only in z-direction

Governing eqn. for stress tensor:

\[
\begin{align*}
\sigma_{xx} + \rho g &= 0 \\
\sigma_{yy} + \rho g &= 0 \\
\sigma_{zz} + \rho g &= 0
\end{align*}
\]

Due to symmetry, only dependence on z-direction

\[
\begin{align*}
\sigma_{xz} &= 0 \\
\sigma_{yz} &= 0 \\
\sigma_{zz} &= 0 \\
\frac{\partial \sigma_{zz}}{\partial z} + \rho g &= 0
\end{align*}
\]

Final set of governing eqns. for stress tensor

(1) \[ \frac{\sigma_{zz}}{\partial z} + \rho g = 0 \]

(note: \( g_z = g \))
Step 2: Governing equations (cont’d)

Link between stress and strain

Linear isotropic elasticity (considering that there is only one nonzero coefficient in the strain tensor, $\varepsilon_{zz}$):

$$\sigma_{11} = \left( K - \frac{2}{3} G \right) \varepsilon_{33}$$

$$\sigma_{22} = \left( K - \frac{2}{3} G \right) \varepsilon_{33}$$

$$\sigma_{33} = \left( K + \frac{4}{3} G \right) \varepsilon_{33} \quad (2)$$

Now combine eqns. (*), (1) and (2):

Substitute (2) in (1):

$$\frac{\partial \varepsilon_{zz}}{\partial z} \left( K + \frac{4}{3} G \right) + \rho g = 0 \quad (4)$$

Substitute (*) in (4):

$$\frac{\partial^2 \xi_z}{\partial z^2} \left( K + \frac{4}{3} G \right) + \rho g = 0$$

$$\frac{\partial^2 \xi_z}{\partial z^2} = -\frac{\rho g}{K + \frac{4}{3} G} \quad (5)$$

$$\frac{\sigma_{zz}}{\partial z} = 0 \quad \frac{\sigma_{yz}}{\partial z} = 0 \quad (6)$$

Step 2 results in a set of differential eqns.
Step 3: Solve governing eqns. by integration

From (5):

\[
\frac{\partial \xi_z}{\partial z} = -\frac{\rho g}{K + \frac{4}{3} G} z + C_1 = \varepsilon_{zz} \quad \text{(first integration)}
\]

\[
\sigma_{zz} = \left( K + \frac{4}{3} G \right) \left( -\frac{\rho g}{K + \frac{4}{3} G} z + C_1 \right) \quad \text{(knowledge of strain enables to calculate stress via eq. (2))}
\]

\[
\xi_z = -\frac{1}{2} \frac{\rho g}{K + \frac{4}{3} G} z^2 + C_1 z + C_2 \quad \text{(second integration)}
\]

From (6):

\[
\frac{\sigma_{xz}}{\partial z} = 0 \quad \frac{\sigma_{yz}}{\partial z} = 0 \quad \rightarrow \quad \sigma_{xz} = \text{const.} = C_3 \quad \sigma_{yz} = \text{const.} = C_4
\]

Solution expressed in terms of integration constants \( C_i \)

Step 4: Apply BCs

**Stress boundary conditions:** Integration provided that

\[
\sigma_{xz} = \text{const.} = C_3 \quad \sigma_{yz} = \text{const.} = C_4
\]

Stress vector at the boundary of the domain:

\[
\vec{T}^d (\vec{n} = -\vec{e}_z, z = 0) = p \vec{e}_z \quad \vec{T} (\vec{n} = -\vec{e}_z, z = 0) = \sigma (z = 0) \cdot (-\vec{e}_z)
\]

Left and right side must be equal, therefore:

\[
\sigma_{xz} = C_3 = 0, \sigma_{yz} = C_4 = 0 \quad \sigma_{zz} = -p \quad \text{Note: Orientation of surface and C.S.}
\]
Step 4: Apply BCs (cont'd)

Further,

\[ \sigma_{zz} = K + \frac{4}{3} G \left( - \frac{\rho g}{K + \frac{4}{3} G} z + C_1 \right) \]  

(general solution)

\[ \sigma_{zz}(z = 0) = C_1 \left( K + \frac{4}{3} G \right) = -p \]  

(at z=0, see previous slide)

This enables us to determine the constant \( C_1 \)

\[ C_1 = - \frac{p}{K + \frac{4}{3} G} \]

Step 4: Apply BCs (cont'd)

Displacement boundary conditions:

\[ \xi_z(z = H) = - \frac{1}{2} \frac{\rho g}{K + \frac{4}{3} G} H^2 - \frac{p}{K + \frac{4}{3} G} H + C_2 \]

Displacement is known at \( z = H \).

\[ \xi_z(z = H) = - \frac{1}{2} \frac{\rho g}{K + \frac{4}{3} G} H^2 - \frac{p}{K + \frac{4}{3} G} H + C_2 = 0 \]

This enables us to determine the constant \( C_2 \)

\[ C_2 = \frac{1}{K + \frac{4}{3} G} \left( \frac{\rho g}{2} H^2 + pH \right) \]
Final solution (summary): Displacement field, strain field, stress field

\[
\begin{align*}
\xi_z(z) &= \frac{1}{4} \left( \frac{\rho g}{2} \left( H^2 - z^2 \right) - p(z - H) \right) \\
\varepsilon_{zz}(z) &= -\frac{\rho g z + p}{K + \frac{4}{3}G} \\
\sigma_{zz}(z) &= -\rho g z + p
\end{align*}
\]