1.050 Engineering Mechanics I

Lecture 25:
Beam elasticity – problem solving technique and examples

Handout

1.050 – Content overview

I. Dimensional analysis
1. On monsters, mice and mushrooms
2. Similarity relations: Important engineering tools

II. Stresses and strength
3. Stresses and equilibrium
4. Strength models (how to design structures, foundations.. against mechanical failure)

III. Deformation and strain
5. How strain gages work?
6. How to measure deformation in a 3D structure/material?

IV. Elasticity
7. Elasticity model – link stresses and deformation
8. Variational methods in elasticity

V. How things fail – and how to avoid it
9. Elastic instabilities
10. Plasticity (permanent deformation)
11. Fracture mechanics

Lectures 1-3 Sept.
Lectures 4-15 Sept./Oct.
Lectures 16-19 Oct.
Lectures 20-31 Oct./Nov.
Lectures 32-37 Dec.
1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

- Lecture 20: Introduction to elasticity (thermodynamics)
- Lecture 21: Generalization to 3D continuum elasticity
- Lecture 22: Special case: isotropic elasticity
- Lecture 23: Applications and examples
- Lecture 24: Beam elasticity
- Lecture 25: Applications and examples (beam elasticity)
- Lecture 26: Cont’d and closure

V. How things fail – and how to avoid it

Beam bending elasticity

Governed by this differential equation:

\[
\frac{d^4 f_z}{dx^4} = \frac{f_z}{EI}
\]

Integration provides solution for displacement

Solve integration constants by applying BCs

Note:

- \( E \) = material parameter (Young’s modulus)
- \( I \) = geometry parameter (property of cross-section)
- \( f_z \) = distributed shear force (force per unit length)
- \( f_z = pb_0 \) where \( p_0 \) = pressure, \( b \) = thickness of beam in y-direction
4-step procedure to solve beam elasticity problems

- **Step 1:** Write down BCs (stress BCs and displacement BCs), analyze the problem to be solved (read carefully!)
- **Step 2:** Write governing equations for $\xi_z$, $\xi_x$ ...
- **Step 3:** Solve governing equations (e.g. by integration), results in expression with unknown integration constants
- **Step 4:** Apply BCs (determine integration constants)

**Note:** Very similar procedure as for 3D isotropic elasticity problems

Difference in governing equations (simpler for beams)

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**Physical meaning of derivatives of $\xi_z$**

\[
\begin{align*}
\frac{d^4 \xi_z}{dx^4} &= f_z \frac{EI}{d^4 \xi_z} EI = f_z & \text{Shear force density} \\
\frac{d^3 \xi_z}{dx^3} &= -Q_z \frac{EI}{d^3 \xi_z} EI = Q_z & \text{Shear force} \\
\frac{d^2 \xi_z}{dx^2} &= -M_y \frac{EI}{d^2 \xi_z} EI = M_y & \text{Bending moment} \\
\frac{d \xi_z}{dx} &= -\omega_y & \text{Rotation (angle)} \\
\xi_z &= \xi_z & \text{Displacement}
\end{align*}
\]
Step-by-step example

Step 1: BCs
\[
\begin{align*}
\xi_z(0) &= 0 \\
\omega_y(0) &= 0 \\
\xi_z(l) &= 0 \\
M_y(0) &= 0 \\
x = 0 & \\
x = l 
\end{align*}
\]

Step 2: Governing equation
\[
\frac{d^4\xi_z}{dx^4} = \frac{f_z}{EI} \quad \Rightarrow \quad \frac{d^4\xi_z}{dx^4} = -\frac{p}{EI}
\]

\(p\) applied in negative \(z\)-direction

Step 3: Integration
\[
\xi_z = -\frac{p}{EI} x + C_1
\]
\[
\xi_z = -\frac{p}{EI} \frac{x^2}{2} + C_1 x + C_2
\]
\[
\xi_z = -\frac{p}{EI} \frac{x^3}{6} + C_1 \frac{x^2}{2} + C_2 x + C_3
\]
\[
\xi_z = -\frac{p}{EI} \frac{x^4}{24} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4
\]
Step 4: Apply BCs

\[
\begin{align*}
\xi''_2 &= -\frac{p}{EI} x + C_1 = -\frac{Q_z}{EI} \\
\xi''_2 &= -\frac{p}{EI} x^2 + C_1 x + C_2 = \frac{M_y}{EI} \\
\xi''_2 &= -\frac{p}{EI} x^3 + \frac{C_1 x^2}{2} + C_2 x = \frac{-\omega_y}{EI} \\
\xi''_2 &= -\frac{p}{EI} x^4 + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4
\end{align*}
\]

Known quantities are marked

Step 4: Apply BCs (cont’d)

\[
\begin{align*}
\xi_2(0) &= 0 \rightarrow C_4 = 0 \\
\omega_y(0) &= 0 \rightarrow C_3 = 0 \\
\xi_2(l) &= 0 \rightarrow -\frac{p}{EI} \frac{l^4}{24} + \frac{C_1 l^3}{6} + \frac{C_2 l^2}{2} = 0 \\
M_y(0) &= 0 \rightarrow -\frac{p}{EI} \frac{l^2}{2} + C_1 l + C_2 = 0
\end{align*}
\]

\[
\begin{pmatrix}
\frac{l^3}{6} & \frac{l^2}{2} \\
\frac{l}{2} & 1
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix}
= \frac{p}{EI} \begin{pmatrix}
\frac{l^4}{24} \\
\frac{l^2}{2}
\end{pmatrix}
\]

\[
\begin{align*}
C_1 &= \frac{p}{EI} \frac{5}{l} \\
C_2 &= -\frac{p}{EI} \frac{l^2}{8}
\end{align*}
\]
Solution:

\[
\begin{align*}
Q_z(x) &= p \left( x - \frac{5}{8} l \right) \\
M_y(x) &= p \left( \frac{1}{8} l^2 + \frac{x^2}{2} - \frac{5}{8} lx \right) \\
\omega_y(x) &= \frac{p}{EI} \left( \frac{1}{8} l^2 x + \frac{x^3}{6} - \frac{5}{16} lx^2 \right) \\
\varepsilon_z(x) &= -\frac{p}{EI} \left( \frac{1}{16} l^2 x^2 + \frac{x^4}{24} - \frac{5}{48} lx^3 \right)
\end{align*}
\]