LECTURE #10

1.060 ENGINEERING MECHANICS II

REYNOLDS TRANSPORT THEOREM

\[
\frac{DM}{Dt} = \text{Rate of change of } M \text{ within } V(t) = \frac{\partial}{\partial t} \int m \, dV + \int m \, q_L \, dA + \int m \, q_R \, dA =
\]

Rate of change of \( M \) between fixed inflow & outflow Sections - Rate of inflow of \( M \) + Rate of outflow of \( M \)

\[
\begin{align*}
\text{Inflow} & \quad + \quad V(t) \\
\bigarrow{\vec{n}} & \quad \bigarrow{\vec{q}} \\
\text{Outflow} & \quad \bigarrow{-\vec{n}} \quad \bigarrow{\vec{q}}
\end{align*}
\]

\( q_L = -\vec{n} \cdot \vec{q} \quad q_T = \vec{n} \cdot \vec{q} = 0 \quad q_R = \vec{n} \cdot \vec{q} \)

CONSERVATION OF (LINEAR) MOMENTUM: \( m = \vec{m} = \vec{q} \)

\[
\frac{DM}{Dt} = \frac{\partial}{\partial t} \int \vec{q} \, dV + \int \vec{q} \left( \vec{n} \cdot \vec{q} \right) \, dS = \sum \text{Forces on } V(t) = \int \vec{q} \, dV + \int \left( -\vec{p} \vec{n} + \vec{\tau}_s \right) \, dS
\]

Gravity Force + Pressure & Shear Forces (\( \vec{\tau}_s \)) on fluid in \( V(t) \) from surrounding fluid and/or boundaries.
CONSERVATION OF MOMENTUM

\[ \frac{D}{Dt} \left( \int_{V(t)} \rho \vec{q} \, dV \right) \] = Rate of change of momentum for material volume

\[ \frac{\partial}{\partial t} \left( \int_{V_0} \rho \vec{q} \, dV \right) + \int_{S_0} \rho \vec{q} \, (\vec{q} \cdot \vec{n}) \, dS = \]

Rate of change in fixed volume + Net rate of outflow from fixed volume =

\[ \int_{V_0} \rho \vec{q} \, dV + \int_{S_2} (-p \vec{n}) \, dS + \int_{S_0} \vec{t} \, dS \]

Gravity + Pressure + Shear
Pick flow areas where flow is well defined.

1) Straight, parallel streamlines with \( A_f \) \perp streamlines.

\[
\frac{\partial}{\partial t} \int_{V_0} p \vec{q} \, dV + \int_{A_f} p \vec{q} \cdot \hat{n} \, dA = \int_{A_f} \vec{F} \cdot \vec{q} \, dA + \int_{A_f} (\hat{n} \cdot \vec{F} + \vec{F}_s) \, dA
\]

\[
\int_{A_f} p \vec{q} \cdot \hat{n} \, dA = \int_{A_f} p q_z \hat{n} \, dA \quad \text{over} \ A_f \text{ whether In- or Outflow.}
2) Well behaved flow ~ little to no shear in velocity across \( A_f \), i.e.,
\[
\int \frac{\overline{v}}{A_f} dA = 0
\]

3) Well behaved flow = Pressure varies hydrostatically \( \perp \) to streamlines (i.e., pressure distribution varies linearly over \( A_f \)).

\[
\int_{A_f} - p \hat{n} dA = -\int_{A_f} p dA \hat{n} = -P_c \cdot \overline{A_f} \cdot \overline{n}
\]

\( P_c \) = Pressure at center of gravity of fluid area 
\( P_c \cdot \overline{A_f} \) = Total pressure force on \( A_f \) on fluid in \( \overline{n} \) from surrounding fluid outside
Pressure force is \( \perp \) \( A_f \) and acts Inwards, i.e., towards \( \overline{n} \) \( [\overline{n} \text{-direction}] \) if \( P_c > 0 \).

**Now the Momentum Equation is**

\[
\frac{\partial}{\partial t} \int_{\Omega} \rho \overline{q} \, dV = \text{Rate of change of momentum within} = \\
\int_{\Omega} \rho \overline{v} \, dV - \int_{A_f} (\rho \overline{v}^2 + p) \, dA_f + \int_{A_s} (-p \hat{n} + \overline{E}_s) \, dA_s = \\
\text{THRUST on flow} + \text{Gravity force \& areas, acting \& forces acting on inwards towards } \overline{n} \text{ \& } \Omega \text{ to } \overline{A_f}'s \text{ fluid within } \Omega
THE THRUST

\[ \text{Thrust} = \left[ \oint \left( \sigma q_l^2 + p \right) dA \right] [-\vec{n}] \]

\[ \int p dA = p_{CG} A = P(\text{resistance}) \text{ force - just like hydrostatics once flow is wellbehaved} \]

\[ p_{CG} = \text{pressure at CG of } A. \]

\[ \oint q_l^2 dA = K_m \sigma V^2 A = K_m \rho V Q \]

\[ K_m = \text{Momentum Coefficient} = \frac{\oint q_l^2 dA}{V^2 A} \approx 1 \text{ if } q_l = V \text{ over } A \]

\[ \oint q_l^2 dA = \oint (V + \vec{q}_l)^2 dA \approx \oint (V^2 + 2Vq_l + q_l^2) dA = \]

\[ V^2 \oint [1 + \left( \frac{q_l}{V} \right)^2] dA \]

\[ \text{If } q_l/V \ll 1 \text{ over most of } A. \]

\[ \oint q_l^2 dA = K_m \sigma V^2 A = M(\text{momentum}) \text{ force} \]

\[ \overrightarrow{\text{THrust}} = \overrightarrow{MP} = (K_m \sigma V^2 A + p_{CG} A)(-\vec{n}) \]

\[ \overrightarrow{MP} \text{ acts perpendicular to wellbehaved flow area and is always directed inwards} \]

\[ \text{towards } \theta_0, \text{ regardless of in- or out-flow and no need to worry about sign of } q_l. \]
THE MOMENTUM PRINCIPLE

\[ \frac{\partial}{\partial t} \int_{V} \rho \vec{v} \, dV = \int_{\partial V} \rho \vec{q} \cdot d\vec{A} + \sum M \vec{P} + (\text{All other forces}) \]

If flow is steady \( \partial / \partial t = 0 \) and

Gravity force \( \int \rho g \, dV \) + Thrusts at flow areas

\( \sum M \vec{P} + \text{Sum of all other forces on fluid in} \ T_0 = 0 \)

Since "Sum of forces on" = "Sum of forces from"

Sum of all forces from fluid in \( T_0 \) on its surroundings (including frictional forces!) =

Gravity force on fluid inside \( T_0 \) +

\( \sum \text{Thrusts that depend only on conditions at inflow and outflow sections to} \ T_0 \)

Powerful Stuff = THE ORIGINAL "BLACK BOX"