If, in addition to a uniform approach flow \( V \), the fluid is "spinning", e.g. induced by the object itself rotating around an axis and dragging the surrounding fluid along, the combination of the two velocity fields will, as illustrated above, force more of the approach flow to pass below the cylinder. This breaks up the symmetry of the flow around the cylinder, and results in larger velocity at \((x, -y)\) than at \((x, y)\). Large velocity gives, according to Bernoulli, lower pressure. Therefore, \( p(x, -y) < p(x, y) \) and upon integration a net downward force, i.e. \(-F \times V\) and therefore a "Lift" force, \( F_L \), results.

In general and in analogy with drag force where

\[
F_L = \frac{1}{2} \rho C_L A L V^2
\]

where \( C_L = \text{lift coefficient}. \)
Working the Angle

\[ \tan \alpha_r = \frac{V_p}{V} \]

\[ \alpha_r < \alpha_p: \]
Wind-Induced \( V_p \)

\[ \alpha_r > \alpha_p: \]
Wind-Resisted \( V_p \)

Working with the Wind

\[ V F_p > 0 \]
We gain Power

Working against the Wind

\[ V \approx 0 \]
\[ V_p F_p < 0 \]
We supply Power
Working the Angle

\[ \tan \alpha_r = \frac{V_p}{V} \]

\( \alpha_r < \alpha_p \):
Wind-Induced \( V_p \)

\( \alpha_r > \alpha_p \):
Wind-Resisted \( V_p \)

Working with the Wind

\( V F_p > 0 \)
We gain Power

Working against the Wind

\( V F_p < 0 \)
We supply Power
Turbo-Power

Side-view

Front-view

Fan

Windmill

$F_T$ against $V_p$

Power required to drive blade to produce flow

$F_T$ with $V_p$

Flow required to drive blade to produce power
Ideal Wind Turbine

\[ \rho = 1.2 \text{ kg/m}^3 \]

Wind Speed

\[ F_{\text{ideal}} = (p_f - p_b) A = \rho V^2 A = \frac{4}{9} \rho V_{\text{in}}^2 \pi R^2 \]

Ideal Power Production

\[ P_{\text{ideal}} = \frac{F_{\text{ideal}} V}{\rho V^3 A} = \frac{8}{27} \rho V_{\text{in}}^3 \pi R^2 \]

Reality

Real Force = \( F < F_{\text{ideal}} \) \hspace{1cm} F \approx 0.7 F_{\text{ideal}}

Real Power = \( P < P_{\text{ideal}} \) \hspace{1cm} P \approx 0.5 P_{\text{ideal}}
Ideal Wind Turbine Theory

Conservation of mass for stream tube

\[ \rho V_{in} A_{in} = \rho V A = \rho V_{out} A_{out} \quad (1) \]

when

\[ \rho = \text{constant} \]

\[ V_{in} A_{in} = VA = V_{out} A_{out} \quad (2) \]

Assumption A: No flow across stream tube walls.
Conservation of Momentum

\[ \rho V_{in}^2 A_{in} = \rho V_{out}^2 A_{out} + F \]  
(3)

or, using (2)

\[ F = \rho (VA)(V_{in} - V_{out}) = \rho V (V_{in} - V_{out}) A \]  
(4)

Assumption B: Pressure forces on steam tube walls and end sections balance out

Assumption C: No momentum inflow or outflow across steam tube walls.

Conservation of Energy

Bernoulli from inflow to "b" (before turbine)

\[ \frac{1}{2} \rho V_{in}^2 + p_o = \frac{1}{2} \rho V_{b}^2 + p_b \]  
(5)

Bernoulli from outflow to "a" (after turbine)

\[ \frac{1}{2} \rho V_{out}^2 + p_o = \frac{1}{2} \rho V_{a}^2 + p_a \]  
(6)

but mass conservation across turbine, with \( A_b \approx A_a = A \) gives

\[ V_b = V_a = V \]  
(7)
So, with (7) we may subtract (6) from (5) to obtain

\[ p_b - p_a = \frac{1}{2} \rho (V_{in}^2 - V_{out}^2) = \frac{1}{2} \rho (V_{in} - V_{out}) (V_{in} + V_{out}) \] (8)

Assumption D: No headloss between inflow and turbine.

Assumption E: No headloss between turbine and outflow.

**Velocity through Turbine**

From (7) it follows that

\[ F = (p_b - p_a) A = \frac{1}{2} \rho (V_{in} - V_{out}) (V_{in} + V_{out}) A \] (9)

by use of (8). Comparison of (9) and (4) then gives

\[ V = \frac{1}{2} (V_{in} + V_{out}) \] (10)

**Power Loss Through Turbine**

Again, accepting (7), we have

\[ P = \dot{E}_b - \dot{E}_a = (p_b - p_a) VA = F V = \frac{1}{2} \rho (V_{in} - V_{out}) (V_{in} + V_{out}) \]

\[ = \frac{1}{4} \rho V_{in}^3 A \left[ 1 - \frac{V_{out}}{V_{in}} \right] \left[ 1 + \frac{V_{out}}{V_{in}} \right] \] (11)
To maximize the power loss of the flow, which equals the power input to the turbine, we take \( V_{\text{out}} / V_{\text{in}} = \alpha \), to get from (11)

\[
P \propto (1-\alpha)(1+\alpha)^2
\]

\[
\frac{dP}{d\alpha} = (1-\alpha)2(1+\alpha) - (1+\alpha)^2 = -3\alpha^2 - 2\alpha + 1 = 0
\]

or

\[
\alpha = V_{\text{out}}/V_{\text{in}} = \frac{1}{3} \quad \text{for} \quad P = P_{\text{max}}
\]  

(12)

Thus, we have for maximum power output from (11)

\[
P_{\text{max}} = \frac{1}{4} \rho V_{\text{in}}^3 A \cdot \frac{2}{3} \left( \frac{4}{3} \right)^2 = \frac{8}{27} \rho V_{\text{in}}^3 A
\]

or, since \( V_{\text{in}} \) is the wind speed and \( A = \pi R^2 \) is the area covered by the turbine blades, we have

\[
P_{\text{max}} = \frac{16}{27} \left( \frac{1}{2} \rho V_{\text{in}}^2 \right) (V_{\text{in}} \pi R^2)
\]

(13)

i.e. \( 16/27 = 0.593 \) [known as the Betz Number] times the rate of kinetic energy transported through area of the turbine's sweep.

Corresponding to (13), i.e. \( P_{\text{max}} \), the force on the turbine is from (91)

\[
F(P_{\text{max}}) = \frac{1}{2} \rho V_{\text{in}}^2 A \left(1-\frac{1}{3}\right)(1+\frac{1}{3}) = \frac{4}{5} \rho V_{\text{in}}^2 (\pi R^2)
\]

(14)