Internal gravity waves (non-rotating)

Boussinesq

From $\rho = \rho_0(1 - b/g)$, $p = -\rho_0gz + \rho_0P$, and $b \ll g$ (and $c_s >> \sqrt{gH}$)

$$\frac{D}{Dt}u = -\nabla P + \dot{zb}$$
$$\nabla \cdot u = 0$$

$$\frac{D}{Dt}b = 0$$

(igw.1)

Linearized

Split into a static state $\frac{\partial^2 b}{\partial z^2} = N^2$, $P = \int^z b$ and deviations. Assume all products of deviations are negligible.

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}P$$
$$\frac{\partial}{\partial t}v = -\frac{\partial}{\partial y}P$$

$$\frac{\partial}{\partial t}w = -\frac{\partial}{\partial z}P + b$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial t}b + wN^2 = 0$$

(igw.2)

Wave solutions

If all fields are proportional to $\exp(ikx + ily + mz - \omega t)$ then

$$\begin{pmatrix}
  -\omega & 0 & 0 & ik & 0 \\
  0 & -\omega & 0 & il & 0 \\
  0 & 0 & -\omega & im & -1 \\
  ik & il & im & 0 & 0 \\
  0 & 0 & N^2 & 0 & -i\omega
\end{pmatrix}
\begin{pmatrix}
  u \\
  v \\
  w \\
  P \\
  b
\end{pmatrix} = 0$$

Nontrivial solutions exist when

$$\omega \left[ \omega^2(k^2 + \ell^2 + m^2) - N^2(k^2 + \ell^2) \right] = 0$$

The $\omega = 0$ root corresponds to geostrophic (here, just horizontally nondivergent), hydrostatic balance; we'll return to that later. The other root

$$\omega = \pm N \left[ \frac{k^2 + \ell^2}{k^2 + \ell^2 + m^2} \right]^{1/2} = \pm N \cos \phi$$

(igw.3)
Fields

If we pick the structure of one field, we can find the others. For \( \theta = k \cdot x - \omega t \), we have

\[
\begin{align*}
  b &= B \cos \theta \\
  w &= -\frac{\omega}{N^2} B \sin \theta \\
  p &= \frac{m}{|k|^2} B \sin \theta \\
  u &= \frac{mk}{\omega |k|^2} B \sin \theta \\
  v &= \frac{ml}{\omega |k|^2} B \sin \theta
\end{align*}
\]

The dispersion relation follows from the continuity eqn.

\[
\frac{mk^2 + ml^2}{\omega |k|^2} - \frac{\omega m}{N^2} = 0
\]

Single variable version

We can eliminate variables in favor of a single field, \( w \), to get the analogue to the classical wave equation. We begin with the divergence of the momentum equations which gives a diagnostic eqn. for the pressure

\[
\nabla^2 P = \frac{\partial}{\partial z} b
\]

which we can use to eliminate \( P \) from the vertical momentum equation

\[
\frac{\partial}{\partial t} \nabla^2 w = -\frac{\partial^2}{\partial z^2} b + \nabla^2 b = \nabla_h^2 b
\]

This gives

\[
\frac{\partial^2}{\partial t^2} \nabla^2 w = -N^2 \nabla_h^2 w \quad (igw.4)
\]

after using the buoyancy equation.
Vorticity version

In the simplest 2D case \((x, z)\), we can write the equation for the \(y\) component of the vorticity, \(\xi = \frac{\partial}{\partial x} u - \frac{\partial}{\partial z} w\),

\[
\frac{\partial}{\partial t} \xi = -\frac{\partial}{\partial x} b
\]

and use the \(y\) component of the streamfunction \(\psi\)

\[
u = \frac{\partial}{\partial z} \psi \quad , \quad w = -\frac{\partial}{\partial x} \psi \quad \Rightarrow \quad \xi = \nabla^2 \psi
\]

to find

\[
\frac{\partial}{\partial t} \nabla^2 \psi = -\frac{\partial}{\partial x} b
\]

\[
\frac{\partial}{\partial t} b = N^2 \frac{\partial}{\partial x} \psi
\]

Taking another time-derivative of the vorticity equation gives (4) written in terms of \(\psi\).

Fourier solution

If we have an initial condition \(w(x, 0)\), we can solve for later times by finding the transform

\[
w(x, t) = \int d^3k \hat{w}(k, t) \exp(i k \cdot x)
\]

Substituting in the dynamical equation (4) gives

\[
\frac{\partial^2}{\partial t^2} \hat{w} = -N^2 \frac{k^2 + \ell^2}{k \cdot k} \hat{w} = -\Omega^2 \hat{w}
\]

which has solutions like

\[
\hat{w}(k, t) = \hat{w}(k, 0) \exp(-i \Omega(k)t)
\]

(There’s also the negative sign case – determining how much of each we have will depend on the initial conditions on \(w\) and \(\frac{d}{dt} w\). We’ll begin assuming those are set so that we have energy in only the positive frequency root.) Then

\[
w(x, t) = \int d^3k \hat{w}(k, 0) \exp(i k \cdot x - i \Omega(k)t)
\]

Demos, Page 3: IGW solutions \(<k=2,m=1,cp=0.4>\quad <k=1,m=2,cp=0.2>\quad <k=2,m=1, cp=0.4, c 0.18>\quad <k=1,m=2,cp=0.2, cg=0.36,-0.18>\quad <k=2,m=2,cp=0.25, cg=0.18,-0.18>\)
Phase and group velocities

The phase of the wave is \( \theta = \mathbf{k} \cdot \mathbf{x} - \omega t \), and the rate of movement in the direction parallel to the wavenumber vector is

\[
\delta \theta = \mathbf{k} \cdot \hat{\mathbf{k}} c \delta t - \omega \delta t = 0 \quad \Rightarrow \quad \delta t \left[ |\mathbf{k}| c - \omega \right] = 0
\]

so that

\[
c = \frac{\omega}{|\mathbf{k}|}
\]

But the packet doesn’t propagate like that at all. To see how it does, let’s suppose the initial condition has a sharply-peaked spectrum

\[
\hat{w}(\mathbf{k}, 0) = \epsilon^{-3} \phi\left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon}\right)
\]

so that the initial condition represents a large-scale modulation of a small-scale wave

\[
w(\mathbf{x}, t) = \int d^3 \mathbf{k} \epsilon^{-3} \phi\left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon}\right) \exp(i \mathbf{k} \cdot \mathbf{x})
\]

\[
= \int d^3 \mathbf{K} \phi(\mathbf{K}) \exp(i \mathbf{k}_0 \cdot \mathbf{x} + i \epsilon \mathbf{K} \cdot \mathbf{x})
\]

\[
= \exp(i \mathbf{k}_0 \cdot \mathbf{x}) \int d^3 \mathbf{K} \phi(\mathbf{K}) \exp(i \mathbf{K} \cdot \epsilon \mathbf{x})
\]

\[
= A(\epsilon \mathbf{x}) \exp(i \mathbf{k}_0 \cdot \mathbf{x})
\]

The time-dependent solution is

\[
w(\mathbf{x}, t) = \int d^3 \mathbf{k} \epsilon^{-3} \phi\left(\frac{\mathbf{k} - \mathbf{k}_0}{\epsilon}\right) \exp(i \mathbf{k} \cdot \mathbf{x} - \Omega(\mathbf{k}) t)
\]

\[
= \int d^3 \mathbf{K} \phi(\mathbf{K}) \exp(i \mathbf{k}_0 \cdot \mathbf{x} + i \epsilon \mathbf{K} \cdot \mathbf{x} - i \Omega(\mathbf{k}_0 + \epsilon \mathbf{K}) t)
\]

\[
= \exp(i \mathbf{k}_0 \cdot \mathbf{x} - i \Omega(\mathbf{k}_0) t) \int d^3 \mathbf{K} \phi(\mathbf{K}) \exp(i \mathbf{K} \cdot \epsilon \mathbf{x} - i \Omega(\mathbf{k}_0 + \epsilon \mathbf{K}) t + i \Omega(\mathbf{k}_0) t)
\]

\[
\approx \exp(i \mathbf{k}_0 \cdot \mathbf{x} - i \Omega(\mathbf{k}_0) t) \int d^3 \mathbf{K} \phi(\mathbf{K}) \exp(i \mathbf{K} \cdot \epsilon \mathbf{x} - i \mathbf{K} \cdot \nabla_k \Omega(\mathbf{k}_0) \epsilon t)
\]

\[
= A(\epsilon |\mathbf{x} - \nabla_k \Omega(\mathbf{k}_0) t|) \exp(i \mathbf{k}_0 \cdot \mathbf{x} - i \Omega(\mathbf{k}_0) t)
\]

Thus the envelope propagates at the group velocity

\[
c_g = \nabla_k \Omega|_{\mathbf{k}=0} \quad or \quad c_{g_i} = \frac{\partial \Omega}{\partial k_i}
\]

For internal waves, the group velocity can be found from

\[
\Omega^2 = \frac{N^2(k^2 + \ell^2)}{k^2 + \ell^2 + m^2} = \frac{N^2 k^2 + \ell^2}{|\mathbf{k}|^2}
\]
giving
\[ 2\Omega \frac{\partial \Omega}{\partial k} = 2 \frac{km^2N^2}{|k|^4} \implies \frac{\partial \Omega}{\partial k} = N \frac{km^2}{|k|^3 \sqrt{k^2 + \ell^2}} \]
and
\[ \frac{\partial \Omega}{\partial \ell} = N \frac{\ell m^2}{|k|^3 \sqrt{k^2 + \ell^2}} , \quad \frac{\partial \Omega}{\partial m} = -N \frac{m \sqrt{k^2 + \ell^2}}{|k|^3} \]
Thus
\[ \mathbf{c}_g = \frac{N}{|k|^3 \sqrt{k^2 + \ell^2}} (km^2, \ell m^2, -m(k^2 + \ell^2)) \]
Note that \( \mathbf{k} \cdot \mathbf{c}_g = 0 \); the group velocity is parallel to the planes of constant phase.

**Vertical modes**

If \( N \) is constant, and the domain is bounded by a bottom and top at \( z = 0 \) and \( z = H \), respectively, we can write
\[ w = W(x, y, t) \sin(M\pi/H) \]
and (for \( M = 1 \))
\[ \frac{\partial^2}{\partial t^2} \left( 1 - \frac{H^2}{\pi^2} \nabla_h^2 \right) W = +N^2 \frac{H^2}{\pi^2} \nabla_h^2 W \]
giving
\[ \omega^2 = \frac{N^2K^2}{1 + K^2} \]
with \( K = kH/\pi \).

Note that the phase speed
\[ c = \frac{N}{\left[ \frac{M^2 \pi^2}{H^2} + k^2 \right]^{1/2}} = \frac{NH}{M\pi} \left[ 1 - \left( \frac{kH}{M\pi} \right)^2 \right]^{-1/2} \]
gives only weak dispersion for long waves \( \delta c/c \sim k^2H^2/M^2\pi^2 \). The long wave speed decreases for modes with more wiggles in the vertical. **Dispersion relation** <disp rel>
Another view of group velocity

Consider superimposing two waves,

$$0.5 \cos(k_1 x - \omega_1 t) + 0.5 \cos(k_2 x - \omega_2 t)$$

with $k_1 < k_2$; the result has a “beat-frequency” modulation. The waves will be back in phase at both their peaks when

$$(N + 1) \frac{2\pi}{k_2} = N \frac{2\pi}{k_1} \Rightarrow N = \frac{k_1}{k_2 - k_1} \Rightarrow L = \frac{2\pi}{k_2 - k_1}$$

To calculate the motion consider the time at which the two peaks catch up with each other

$$X_1 = -\frac{2\pi}{k_1} + \frac{\omega_1}{k_1}T = X_2 = -\frac{2\pi}{k_2} + \frac{\omega_2}{k_2}T \Rightarrow T = \frac{2\pi(k_2 - k_1)}{\omega_1 k_2 - \omega_2 k_1}$$

and we see a new maximum constructive interference point at $X_1$. The speed of motion is therefore

$$\frac{X_1}{T} = \frac{\omega_1}{k_1} - \frac{2\pi}{k_1T} = \frac{\omega_2 - \omega_1}{k_2 - k_1}$$

Demos, Page 6: Two waves <initial> <initial> Demos, Page 6: Two waves <initial> <initial> <initial> <initial> <initial> <initial> <initial> <initial> <initial>
Revision of two waves

Assuming that the shorter wave travels more slowly and the two waves start at time \( t = 0 \) being in phase at \( x = 0 \), the pattern will repeat exactly when the previous crests of each of the two waves match up precisely:

\[
\begin{align*}
  c_1 T &= \frac{2\pi}{k_1} + c_g T, \\
  c_2 T &= \frac{2\pi}{k_2} + c_g T
\end{align*}
\]

We can think of these as simultaneous equations for \( 1/T \) and \( c_g \):

\[
\begin{align*}
  c_g + \frac{2\pi}{k_1} \frac{1}{T} &= c_1 \\
  c_g + \frac{2\pi}{k_2} \frac{1}{T} &= c_2
\end{align*}
\]

solving these gives

\[
 c_g = \frac{c_1 \frac{2\pi}{k_2} - c_2 \frac{2\pi}{k_1}}{\frac{2\pi}{k_2} - \frac{2\pi}{k_1}} = \frac{c_1 k_1 - c_2 k_2}{k_1 - k_2} = \frac{\omega_1 - \omega_2}{k_1 - k_2}
\]
Dispersion of group

If we consider the next order in our expansion for sharply peaked spectra

\[ w \simeq \exp(i\mathbf{k}_0 \cdot \mathbf{x} - i\Omega(k_0)t) \int d^3\mathbf{K} \phi(\mathbf{K}) \exp(i\mathbf{K} \cdot \epsilon \mathbf{x} - i\mathbf{K} \cdot \nabla \Omega(k_0)ct) \times \]

\[ \exp(-i\frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \epsilon^2 t) \]

In a frame moving with the group \( \mathbf{x} = \epsilon \mathbf{x} - \mathbf{c}_g ct \), the changes on a time scale \( \tau = \epsilon^2 t \) are determined by

\[ w \simeq \exp(i\mathbf{k}_0 \cdot \mathbf{x} - i\Omega(k_0)t) \int d^3\mathbf{K} \phi(\mathbf{K}) \exp(i\mathbf{K} \cdot \mathbf{X}) \exp\left(-i\frac{1}{2} \frac{\partial^2 \Omega}{\partial k_i \partial k_j} K_i K_j \tau \right) \]

For this we find the amplitude satisfies the Schrödinger equation

\[ \frac{\partial}{\partial \tau} w = i \frac{\partial^2 \Omega}{2 \partial k_i \partial k_j} \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} w \]

This looks like a diffusion equation (with an imaginary diffusivity) and can be solved in much the same way. In particular, we can look for Gaussian solutions

\[ w = A(\tau) \exp(-\alpha_{ij}(\tau) X_i X_j) \]

The result can be seen in the 1D case

\[ \frac{\partial}{\partial \tau} w = i \frac{\partial^2 \Omega}{2 \partial k^2} \frac{\partial^2}{\partial X^2} w \]

Plugging \( w = A(\tau) \exp(-\alpha(\tau) X^2) \) into the previous equation and gathering the terms which are proportional to \( X^2 \) and to 1 gives

\[ \frac{\partial}{\partial t} \alpha = -2i\Omega'' \alpha^2 \]

\[ \frac{\partial}{\partial t} A = -i\Omega'' \alpha A \]

This has solutions

\[ \alpha = \frac{\alpha_0}{1 + 2i\alpha_0 \Omega'' t} \]

\[ A = A(0) \sqrt{\frac{\alpha(t)}{\alpha_0}} \]

Demos, Page 8: 1D case <packet motion and spread> <amplitude decay>
Energy

We can form an energy equation from the linearized equations

$$\frac{\partial}{\partial t} \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) = \frac{\partial}{\partial t} KE = -\mathbf{u} \cdot \nabla P + wb$$
$$= -\nabla \cdot (\mathbf{u} P) + wb$$

The last term represents generation of kinetic energy from potential energy (heavy fluid \(b < 0\) moving down \(w < 0\) or light fluid moving up). We can define the available potential energy here as \(\frac{1}{2}b^2/N^2\)

$$\frac{\partial}{\partial t} \frac{1}{2} b^2 = \frac{\partial}{\partial t} PE = -wb$$

so that the change in total energy in a parcel is given by the flux across the boundary of the parcel \(\mathbf{u}P\)

$$\frac{\partial}{\partial t} KE + PE = \frac{\partial}{\partial t} E = -\nabla \cdot (\mathbf{u} P)$$

For IGW’s, the energy is

$$E = \frac{1}{2} \left( \frac{m^2(k^2 + \ell^2)}{\omega^2 |k|^4} + \frac{\omega^2}{N^4} \right) B^2 \sin^2 \theta + \frac{1}{2} \frac{1}{N^2} \frac{B^2}{2} \cos^2 \theta$$
$$= \frac{1}{2} \frac{B^2}{N^2} \sin^2 \theta + \frac{1}{2} \frac{B^2}{N^2} \cos^2 \theta$$
$$= \frac{1}{2} \frac{B^2}{N^2}$$

and the flux is just

$$\mathbf{F} = \frac{m}{|k|^2} \left( \frac{mk}{\omega |k|^2}, \frac{m\ell}{\omega |k|^2}, -\frac{\omega}{N^2} \right) B^2 \sin^2 \theta$$

which averages to

$$\mathbf{F} = \frac{1}{2} \frac{m}{|k|^2} \left( \frac{mk}{\omega |k|^2}, \frac{m\ell}{\omega |k|^2}, -\frac{\omega}{N^2} \right) B^2$$
$$= \frac{1}{2} \frac{B^2}{N^2} \left( \frac{m^2 k N}{\sqrt{k^2 + \ell^2} |k|^3}, \frac{m^2 \ell N}{\sqrt{k^2 + \ell^2} |k|^3}, -\frac{m \sqrt{k^2 + \ell^2} N}{|k|^3} \right) B^2$$
$$= c_g E$$

The energy moves at the group velocity.
Reflection from a sloping surface

Demos, Page 9: Reflection problem <geometry>

Let us consider IGW's incident on a surface sloped at angle $\beta$ from horizontal. The boundary condition at the surface $z = z \tan \beta$ is that the normal component of the velocity must vanish

$$\mathbf{u} \cdot \hat{n} = -u \sin \beta + w \cos \beta = 0$$

In terms of the streamfunction, this condition becomes

$$-\sin \beta \frac{\partial \psi}{\partial z} - \cos \beta \frac{\partial \psi}{\partial x} = -\hat{t} \cdot \nabla \psi = 0$$

so that we can simply assume $\psi(s \cos \beta, s \sin \beta, t) = 0$ with $s$ being the distance along the slope.

We write the solution in terms of an incident wave and a reflected wave

$$\psi = A \cos(kx + mz - \omega t) + A_r \cos(k_r x + m_r z - \omega_r t)$$

and apply the boundary condition

$$A \cos(s \mathbf{k} \cdot \hat{t} - \omega t) + A_r \cos(s \mathbf{k}_r \cdot \hat{t} - \omega_r t) = 0$$

we see that $\omega_r = \omega$, $\mathbf{k} \cdot \hat{t} = \mathbf{k}_r \cdot \hat{t}$, and $A_r = -A$ in order that the equation above holds for all $s$ and $t$.

The geometry is now clear: since the frequencies are the same,

$$\cos(\phi_r) = \cos(\phi)$$

so that the angle of the reflected wavenumber from horizontal is the same as that of the incident wave; secondly, the projection along the slope must match.

Demos, Page 10: incident/refl <geometry> <group vel> <phi=30,beta=0> <wavefronts> <phi=30,beta=15> <wavefronts> <phi=30,beta=25> <wavefronts> <phi=30,beta=45> <wavefronts>

Note that the reflected wavenumber satisfies

$$K_r \cos(\phi + \beta) = K_i \cos(\phi - \beta)$$

and becomes infinite when $\phi + \beta = 90^\circ$ — when the reflected wave group velocity is tangent to the slope. Beyond this critical slope, we satisfy the boundary condition by using $\omega_r = -\omega$ and $\mathbf{k} \cdot \hat{t} = -\mathbf{k}_r \cdot \hat{t}$.

Demos, Page 10: large slope <phi=60,beta=15> <wavefronts> <phi=60,beta=45> <wavefronts> <phi=60,beta=55> <wavefronts> <phi=60,beta=85> <wavefronts>

When the slope is shallow, a wave can focus between the surface and the bottom and propagate into the corner, with its scale becoming smaller and smaller. Demos, Page 10: focus <incident> <1st refl> <2nd refl> <3rd refl> <4th refl> <5th refl> <6th refl> <7th refl>
Generation by flow over topography

One mechanism for creating internal gravity wave is flow over topography. We’ll consider the simple case with zonal flow at a sinusoidal topography at \( z = h_0 \cos(kx) \). The equations of motion will be linearized assuming the mean flow, \( U \), is much larger than the wave flows \( u \).

\[
\frac{\partial}{\partial t} u + U \frac{\partial}{\partial x} u = -\nabla P + b \hat{z} \\
\nabla \cdot u = 0
\]

\[
\frac{\partial}{\partial t} b + U \frac{\partial}{\partial x} b + wN^2 = 0
\]

The dispersion relation is the same, except we replace \( \omega \) by \( \omega - kU \)

\[
\omega - kU = \pm \frac{Nk}{|k|}
\]

For the steady response \( \omega = 0 \), we will need to use the minus sign.

\[
\omega = kU - \frac{Nk}{|k|}
\]

The condition at the bottom is, again, no normal flow.

\[
(U \hat{x} + u) \cdot \hat{n} = (U \hat{x} + u) \cdot \frac{\hat{z} - \nabla h}{\sqrt{1 + |\nabla h|^2}} = 0
\]

or

\[
(U + u) \frac{\partial}{\partial x} h = w \quad \text{at} \quad z = h(x, y)
\]

(We can find the normal by thinking about a function \( F(x, y, z) = z - h(x, y) \); its three-dimensional gradient is perpendicular to the surfaces of constant \( F \), in particular the one at \( F = 0 \) which represents the boundary.) This linearizes to

\[
w = U \frac{\partial}{\partial x} h \quad \text{at} \quad z = 0
\]

when the slope and the net height change is small. This can also be written as

\[
\psi(x, 0, t) + U h(x, y) = 0
\]
Steady solution [short scales]

For a steady solution, we have

\[ \sqrt{k^2 + m^2} = \frac{N}{U} \quad \text{or} \quad m^2 = \frac{N^2}{U^2} - k^2 \]

If the topographic scale is short compared to \( U/N \), the \( m^2 \) will be negative so that if \( \hat{m} = \sqrt{k^2 - N^2/U^2} \) then

\[ \psi = -U h_0 \Re(e^{ikx \mp \hat{m}z}) \]

We must choose the negative sign so that the disturbance decays with height

\[ \psi = -U h_0 \cos(kx) \exp(-\sqrt{k^2 - N^2/U^2} z) \]

Demos, Page 11: short scales  \(<Uk/N=1.01> <Uk/N=1.1> gwtop2.png <Uk/N=1.5> <Uk/N=2>\)

Long scales

If \( k^2 < N^2/U^2 \) then \( m \) is real and our solution looks like

\[ \psi = -U h_0 \Re(e^{ikx \pm mz}) \]

and we must decide which sign to use (or have some contribution from each). We shall discuss a number of ways of resolving the issue.

Demos, Page 12: long scales  \(<Uk/N=0.99> <Uk/N=0.9> <Uk/N=0.8> <Uk/N=0.7>\)

Group Velocity: Since the topography is the source of the waves, we would expect the vertical component of \( C_g \) to be positive. This means that if we suddenly add or eliminate the topography, the disturbance in the wave field would propagate upwards. Therefore

\[ \frac{\partial}{\partial m} \left[ U k - \frac{N k}{\sqrt{k^2 + m^2}} \right] = \frac{N k m}{(k^2 + m^2)^{3/2}} > 0 \]

The positive sign is the correct one, so that

\[ \psi = -U h_0 \cos(kx + \sqrt{N^2/U^2 - k^2} z) \]
Energie Flux: For these 2-D motions, we can write the average (as in zonal average) vertical energy flux as
\[
\overline{wP} = -\frac{\partial \psi}{\partial x} P = \psi \frac{\partial P}{\partial x}
\]
and we expect it to be positive. Using the zonal momentum equation gives
\[
\overline{wP} = - \frac{\partial^2 \psi}{\partial t \partial z} - U \frac{\partial^2 \psi}{\partial x \partial z} = - \psi \frac{\partial^2 \psi}{\partial t \partial z} + U \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial z}
\]
For steady flow with \( \psi = -U h_0 \cos(kx \pm mz) \), we have
\[
\overline{wP} = \pm \frac{1}{2} U^3 h_0^2 km
\]
again showing the plus sign to be the desired one.

Damping: Another approach is to add damping to the equations so that even the vertically wavy mode decays and reject any growing solution. We take
\[
\frac{\partial}{\partial t} \mathbf{u} + U \frac{\partial}{\partial x} \mathbf{u} = -\nabla P + b \hat{z} - \epsilon \mathbf{u}
\]
\[
\nabla \cdot \mathbf{u} = 0
\]
\[
\frac{\partial}{\partial t} b + U \frac{\partial}{\partial x} b + w N^2 = -\epsilon b
\]
We now have
\[
(ikU + \epsilon)^2 = -\frac{N^2 k^2}{k^2 + m^2} \quad \Rightarrow \quad m^2 = \frac{N^2}{U^2(1 - \epsilon/kU)^2} - k^2
\]
The imaginary part of \( m \) is
\[
\Im(m) \simeq \frac{1}{\Re(m)} \frac{\epsilon N^2}{kU^3}
\]
so that vertically decaying solutions \( \Im(m) > 0 \) require \( \Re(m) > 0 \) as before.
**Initial Value Problem:** Finally, we can look at what happens if we suddenly turn the flow or the topography on. Using

\[
\left( \frac{\partial}{\partial t} + ikU \right)^2 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \psi = k^2N^2\psi
\]

with the initial and boundary conditions

\[
\psi(z, 0) = 0 \ , \ \psi(0, t) = -Uh_0 \ , \ \psi(\infty, t) = 0
\]

The Laplace transformed problem gives the same \(z\) structure equation as in the damped system

\[
\left( \frac{\partial^2}{\partial z^2} - k^2 \right) \psi^T = -\frac{N^2}{(U^2 - 2isU/k - s^2/k^2)}\psi^T
\]

with

\[
\psi^T(0, s) = -Uh_0/s \ , \ \psi^T(\infty, s) = 0
\]

Again the positive root is the proper one

\[
\psi^T = -Uh_0 \frac{1}{s} \exp(i \left[ \frac{N^2}{(U^2 - 2isU/k - s^2/k^2)} - k^2 \right]^{1/2} z)
\]

The inverse transform

\[
\psi = -Uh_0 \int_{-\infty}^{\infty} \frac{1}{s} \exp(i \left[ \frac{N^2}{(U^2 - 2isU/k - s^2/k^2)} - k^2 \right]^{1/2} z)e^{st}\]

is dominated by the singularity at \(s = 0\); for large time, we recover the standing wave solution.

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Demos, Page 14: IGW data  <Cloud patterns>  <Breaking waves>  <Appalachian>  <Surface slicks>  <Internal tides>  <Georges Bank>  <Northern Oregon>
The Nonlinear Problem

We can also look at the nonlinear problem in simple 2-D cases. The steady equations

\[ u \cdot \nabla q = -\frac{\partial}{\partial x} b \]
\[ u \cdot \nabla (b + N^2 z) = 0 \]

can be solved by noting that \( u \cdot \nabla \phi = 0 \) implies \( \phi = \Phi(\psi) \) - the advected property is constant along streamlines, since the parcels of fluid move along the streamlines in steady flows. The streamfunction here includes both the mean flow and the fluctuations \( \psi = Uz + \psi'(x) \). Therefore

\[ N^2 z + b'(x) = B(Uz + \psi'(x)) = \frac{N^2}{U} (Uz + \psi') \]

Uniqueness could be a problem, of course. In any case, we’ll take

\[ b' = \frac{N^2}{U} \psi' \]

The vorticity equation then tells us that

\[ u \cdot \nabla q = \omega \frac{N^2}{U} \quad \Rightarrow \quad u \cdot \nabla (q - \frac{N^2}{U} z) = 0 \]

so that

\[ \nabla^2 \psi' - \frac{N^2}{U^2} z = Q(Uz + \psi') = -\frac{N^2}{U^2} (Uz + \psi') \]

or

\[ \nabla^2 \psi' = -\frac{N^2}{U^2} \psi' \]

with the boundary conditions

\[ \psi'(x, h) + Uh = 0 \quad , \quad \psi' \to 0 \quad or \quad radiation \ condition \]

Note that the linear solution is a perfectly good one – we just have to find the topography that matches it!

\[ Uh_0 \cos(kx) \exp(-\bar{m}h) = Uh \quad or \quad h_0 \cos(kx) = h \exp(\bar{m}h) \]

\[ h_0 \cos(kx + mh(x)) = h(x) \]

Demos, Page 15: topographies  
\(<Uk/N=1.001>  \quad <Uk/N=1.01>  \quad <Uk/N=1.05>  \quad <Uk/N=1.07>  \quad <Uk/N=1.08>  \quad <Uk/N=0.99>  \quad <Uk/N=0.9>  \quad <Uk/N=0.5>  \quad <Uk/N=0.2> \]
WKB and modes

We will now consider propagation and vertical modes in the case where the Brunt-
Väisälä frequency varies with $z$. The normal mode problem

$$\frac{\partial^2}{\partial t^2} \nabla^2 w + N^2 \nabla^2_h w = 0 \quad , \quad w(0) = w(H) = 0$$

can be separated as $w = W(z) \exp(i[kx + \ell y - \omega t])$, and the eigenvalue problem becomes

$$\frac{\partial^2}{\partial z^2} W - k_h^2 W + \frac{N^2}{\omega^2} k_h^2 = 0 \quad , \quad W(0) = W(H) = 0$$

with $k_h^2 = k^2 + \ell^2$. This is a Sturm-Liouville problem with eigenvalue $\lambda = 1/\omega^2$. For a
given $N^2(z)$ and $k_h$, we will find an infinite set of eigenfunctions with increasing $\lambda$'s and
therefore decreasing frequencies. In the case of constant $N$, we have

$$W = \sin(M\pi z/H) \quad , \quad \omega = N \sqrt{\frac{k_h^2 H^2}{k_h^2 H^2 + M^2 \pi^2}}$$

When $N$ is not constant, we can look for approximate solutions

$$W \simeq A(\epsilon z) \sin(\epsilon^{-1} \theta(\epsilon z))$$

so that

$$\frac{\partial^2}{\partial z^2} W = \epsilon^2 A'' \sin \frac{\theta}{\epsilon} + \epsilon[2A' \theta' + A\theta''] \cos \frac{\theta}{\epsilon} - A \theta'' \sin \frac{\theta}{\epsilon}$$

The lowest order problem gives

$$\theta'' = k_h^2 \left[ \frac{N^2}{\omega^2} - 1 \right]$$

or

$$\theta = |k_h| \int_0^z \sqrt{\frac{N^2}{\omega^2} - 1}$$

and an eigenvalue relationship

$$M\pi = |k_h| H \frac{1}{H} \int_0^H \sqrt{\frac{N^2}{\omega^2} - 1}$$

The amplitude satisfies

$$2A' \theta' + A\theta'' = 0 \quad \Rightarrow \quad 2 \frac{1}{A} \frac{d}{dz'} A = - \frac{1}{\theta'} \frac{d}{dz'} \theta'$$
so that
\[ A = \text{const} \frac{\theta'}{\theta}^{1/2} = \text{const} |k_h|^{-1/2} \left[ \frac{N^2}{\omega^2} - 1 \right]^{-1/4} \]
— the amplitude is largest in the regions of small \( N \) and slowly changing phase. This makes sense since the energy flux is proportional to the vertical wavenumber times the amplitude squared.

The results above assume that \( \omega \) is smaller than the minimum value of \( N \) so that the solution remains sinusoidal. When that is not the case, the waves will have a turning point and will be exponentially damped in the region where \( \omega > N \).

Demos, Page 16: modes \(<N\ squared> <kh=0.01> <kh=0.1> <kh=1> <kh=10> <disp\ rel> <wkb\ disp\ rel>

**More general WKB form**

We now consider propagation in an inhomogeneous medium. The waves will be locally sinusoidal
\[ w = A(x, t) \exp(\theta(x, t)) \]
with the implicit assumption that the gradients of \( \theta \) are large (we could put in the \( \epsilon \) factors as before). Locally, we can identify the effective wavenumber and frequency in terms of the derivatives of \( \theta \):
\[ k = \nabla \theta, \quad \omega = -\frac{\partial}{\partial t} \theta \]

To calculate the evolution of the phase, we note that the largest terms in the dynamical equation will give the dispersion relation
\[ \left( \frac{\partial \theta}{\partial t} \right)^2 |\nabla \theta|^2 - N^2 |\nabla_h \theta|^2 = 0 \]
where \( \nabla_h \) is the horizontal gradient \((\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)\) (in Cartesian coordinates). From this, we obtain an evolution equation for \( \theta \)
\[ \frac{\partial}{\partial t} \theta = -N \frac{|\nabla_h \theta|}{|\nabla \theta|} = -\Omega(\nabla \theta, x, t) \]

This equation implies that the wavenumbers and frequency evolve as the wave packet moves through the medium. To see this, consider the case \( N = N(z) \) and suppose that the horizontal and vertical wavenumbers are initially constant
\[ \theta(x, z, 0) = k_0 x + m_0 z \]

At time zero, we have
\[ \frac{\partial}{\partial t} \theta = -N(z) \frac{k_0}{\sqrt{k_0^2 + m_0^2}} \]
so that
\[ \theta(x, z, \delta t) = k_0 x + m_0 z - N(z) \frac{k_0}{\sqrt{k_0^2 + m_0^2}} \delta t \]
and the vertical wavenumber is now
\[ \frac{\partial}{\partial z} \theta = m_0 - N'(z) \frac{k_0}{\sqrt{k_0^2 + m_0^2}} \delta t \]
giving a new frequency as well. Likewise, even with constant \( N \), if \( \frac{\partial}{\partial z} \theta \) is not constant, the value of \( \frac{\partial}{\partial t} \theta \) will also be non-uniform. Therefore the vertical wavenumber changes. To see this more precisely, we can take a \( z \) derivative of the evolution equation to find
\[ \frac{\partial}{\partial t} \theta_z = - \frac{\partial \Omega}{\partial \theta_z} \frac{\partial \theta_z}{\partial z} - \frac{\partial \Omega}{\partial \theta_z} = -c_g \frac{\partial \theta_z}{\partial z} = - \frac{\partial \Omega}{\partial \theta_z} \]
Using \( \frac{\partial}{\partial z} \theta = \theta_z \). In the internal gravity wave case
\[ \frac{\partial}{\partial t} \theta_z = -N'(z) \frac{k_0}{\sqrt{k_0^2 + \theta_z^2}} + N \frac{k_0 \theta_z}{[k_0^2 + \theta_z^2]^{3/2}} \]
In general, we find
\[ \frac{\partial}{\partial t} \nabla_i \theta + (c_g \cdot \nabla) \nabla_i \theta \equiv D_g \theta_i = -\nabla_i \Omega \]
Likewise, we note that \( \frac{\partial}{\partial t} \frac{\partial \theta}{\partial t} \) at time \( \delta t \) will be different from \( \frac{\partial}{\partial t} \frac{\partial \theta}{\partial t} \) at time 0, since it will be evaluated using the new vertical wavenumber.
\[ \frac{\partial}{\partial t} \frac{\partial \theta}{\partial t} + c_g \cdot \nabla \frac{\partial \theta}{\partial t} = D_g \theta = -\frac{\partial}{\partial t} \Omega \]
Finally, we look at the amplitude equation, which comes from the first order terms in
\[ (i\theta_t + \frac{\partial}{\partial t})^2 (i \nabla \theta + \nabla)^2 A + (i \nabla h \theta + \nabla h)^2 AN^2 = 0 \]
This gives
\[ \theta_t \frac{\partial}{\partial t} (|\nabla \theta|^2 A) + \frac{\partial}{\partial t} (\theta_t |\nabla \theta|^2 A) + \theta_t^2 \nabla \theta \cdot \nabla A + \theta_t^2 \nabla \cdot (A \nabla \theta) - \nabla h \theta \cdot \nabla h (N^2 A) - \nabla h \cdot (N^2 A \nabla h \theta) \]
the terms involving derivatives of \( A \) are
\[ 2 \theta_t |\nabla \theta|^2 \frac{\partial A}{\partial t} + 2 \theta_t |\nabla \theta|^2 \frac{\theta_t \nabla \theta}{|\nabla \theta|^2} \cdot \nabla A - 2 \theta_t |\nabla \theta|^2 \frac{\theta_t \nabla h \theta}{|\nabla h \theta|^2} \cdot \nabla A = 2 \theta_t |\nabla \theta|^2 D_g A \]
using the dispersion relation
\[ \theta^2 = N^2 \frac{|\nabla h \theta|^2}{|\nabla \theta|^2} \]
and the definition of the group velocity, which gives
\[ c_g = \theta_t \left[ \frac{\nabla \theta}{|\nabla \theta|^2} - \frac{\nabla h \theta}{|\nabla h \theta|^2} \right] \]

The amplitude equation becomes
\[ 2\theta_t |\nabla \theta|^2 D_g A + A \left[ 2\theta_t \frac{\partial}{\partial t} |\nabla \theta|^2 + |\nabla \theta|^2 \frac{\partial \theta_t}{\partial t} + \theta_t^2 |\nabla \theta|^2 - \nabla h \theta \cdot \nabla N^2 - \nabla h \cdot (N^2 \nabla h \theta) \right] = 0 \]

Multiplying by \( \frac{1}{2} A^* \) and adding the conjugate gives
\[ \theta_t |\nabla \theta|^2 D_g |A|^2 + |A|^2 \left[ 2\theta_t \frac{\partial}{\partial t} |\nabla \theta|^2 + |\nabla \theta|^2 \frac{\partial \theta_t}{\partial t} + \theta_t^2 |\nabla \theta|^2 - \nabla h \theta \cdot \nabla N^2 - \nabla h \cdot (N^2 \nabla h \theta) \right] = 0 \]

Next we manipulate the divergence terms
\[ \theta_t^2 |\nabla \theta|^2 - \nabla (N^2 \nabla h \theta) = \nabla \cdot \left( \theta_t |\nabla \theta|^2 \frac{\theta_t \nabla \theta}{|\nabla \theta|^2} - \theta_t |\nabla \theta|^2 \frac{\theta_t \nabla h \theta}{|\nabla h \theta|^2} \right) - \nabla \theta \cdot \nabla \theta_t^2 = \nabla \cdot (\theta_t |\nabla \theta|^2 c_g) - \nabla \theta \cdot \nabla \theta_t^2 \]

Combining these with the first two terms gives
\[ 2\theta_t \frac{\partial}{\partial t} |\nabla \theta|^2 + |\nabla \theta|^2 \frac{\partial \theta_t}{\partial t} + \theta_t^2 |\nabla \theta|^2 - \nabla h \cdot (N^2 \nabla h \theta) = \theta_t \frac{\partial}{\partial t} |\nabla \theta|^2 + D_g (\theta_t |\nabla \theta|^2) + \theta_t |\nabla \theta|^2 \nabla \cdot c_g - \nabla \theta \cdot \nabla \theta_t^2 \]

Finally, we use
\[ -2N \nabla h \theta \cdot \nabla h N = 2N \nabla h \theta \cdot \frac{|\nabla \theta|}{|\nabla h \theta|} D_g \nabla h \theta = -\theta_t |\nabla \theta|^2 \frac{1}{|\nabla h \theta|^2} D_g |\nabla h \theta|^2 \]

Putting these all together gives
\[ \theta_t |\nabla \theta|^2 D_g |A|^2 + |A|^2 \left[ D_g \theta_t |\nabla \theta|^2 - \theta_t |\nabla \theta|^2 \frac{1}{|\nabla h \theta|^2} D_g |\nabla h \theta|^2 + \theta_t |\nabla \theta|^2 \nabla \cdot c_g \right] = 0 \]

Multiply everything by \( 1/\theta_t |\nabla \theta|^2 |A|^2 \), put in terms of logs, combine the terms, and undo the logs; we find
\[ D_g \theta_t \frac{|\nabla \theta|^2 |A|^2}{|\nabla h \theta|^2} + \theta_t \frac{|\nabla \theta|^2 |A|^2}{|\nabla h \theta|^2} \nabla \cdot c_g = 0 \]

If we substitute the definition of energy from the lowest order relationships
\[ u \theta_t u = -i \nabla \theta p + b \tilde{u} \]
\[ \nabla \theta \cdot u = 0 \]
\[ u \theta_t b + w N^2 = 0 \]
giving 
\[ E = \frac{\left| \nabla \theta \right|^2}{\left| \nabla_{\theta} \theta \right|^2} |A|^2 \]

Thus we arrive at the form 
\[ \frac{\partial}{\partial t} \theta_t E + \nabla \cdot (c_g \theta_t E) = 0 \]

Using \( Dg \theta_t = 0 \) (\( N^2 \) independent of time) tells us that the energy in the wave changes by 
divergences of the energy flux 
\[ \frac{\partial}{\partial t} E + \nabla \cdot (c_g E) = 0 \]

Obviously, this is a non-trivial process; for the internal gravity wave problem, we can 
take a simpler approach, which is to work directly from the equations of motion assuming 
all the variables have a WKB form: 
\[ b = b(x, t)e^{i\theta(x, t)} + b^*(x, t)e^{-i\theta(x, t)} \]

Then the equations become 
\[ i\theta_t u + \frac{\partial u}{\partial t} = -iP\nabla \theta - \nabla P + b \]
\[ i\nabla \theta \cdot u + \nabla \cdot u = 0 \]
\[ i\theta_t b + \frac{\partial b}{\partial t} + w N^2 = 0 \]

The kinetic energy, averaged over the rapidly varying phase is 
\[ \frac{1}{2} \langle (ue^{i\theta(x, t)} + u^*e^{-i\theta(x, t)})^2 \rangle = \frac{1}{2} \langle u \cdot u e^{2i\theta(x, t)} + 2u \cdot u^* + u^* \cdot u e^{-2i\theta(x, t)} \rangle = u \cdot u^* \]

so that we can form the KE eqn. by dotting the first equation with \( u^* \) and adding the 
conjugate 
\[ \frac{\partial}{\partial t} u \cdot u^* = -i(Pu^* \cdot \nabla \theta - P^* u \cdot \nabla \theta) - u^* \cdot \nabla P - u \cdot \nabla P^* + w^* b + wb^* \]
\[ = -P \cdot \nabla u^* - P^* \nabla \cdot u - u^* \cdot \nabla P - u \cdot \nabla P^* + w^* b + wb^* \]
\[ = -\nabla \cdot (uP^* + u^* P) + w^* b + wb^* \]

(using the continuity equation). Multiplying the buoyancy equation by \( b^* \) and adding the 
conjugate gives 
\[ \frac{\partial}{\partial t} bb^* + (wb^* + w^* b) N^2 = 0 \]

or, in terms of the available potential energy \( bb^*/N^2 \) (assuming \( N^2 \) is time-independent), 
\[ \frac{\partial}{\partial t} \frac{bb^*}{N^2} + wb^* + w^* b = 0 \]

From these, we find 
\[ \frac{\partial}{\partial t} E = -\nabla \cdot (uP^* + u^* P) \]

or, using the equipartition of energy at the lowest order (all that’s now needed) 
\[ \frac{\partial}{\partial t} E = -\nabla \cdot (c_g E) \]