Laplace Tidal Equations — Vertical Structure

The vertical structure is given by the solution of

$$\frac{\alpha}{\partial z} \frac{1}{\alpha N^2} \frac{\partial F}{\partial z} + \left[ \frac{\omega^2}{\alpha^2 (N^2 - \omega^2)} - \frac{\partial}{\partial z} \frac{\omega^2}{g (N^2 - \omega^2)} \right] F = - \frac{1}{g H_{eq}} F$$

with

$$(\frac{\partial}{\partial z} - \frac{N^2}{g}) F = 0 \quad (FixedB)$$

and/or

$$(\frac{\partial}{\partial z} - \frac{\omega^2}{g}) F = 0 \quad (FreeB)$$

Low frequency

When $\omega$ is small, the VSE simplifies to

$$\frac{\alpha}{\partial z} \frac{1}{\alpha N^2} \frac{\partial F}{\partial z} = - \frac{1}{g H_{eq}} F$$

with

$$(\frac{\partial}{\partial z} - \frac{N^2}{g}) F = 0 \quad (FixedB)$$

and/or

$$\frac{\partial}{\partial z} F = 0 \quad (FreeB)$$

In the case where $N^2 H << g$, the boundary conditions become $F_z = 0$ at both types of surfaces. This equation clearly has a barotropic solution $F = 1$ with infinite equivalent depth. If we don’t drop the $N^2 / g$ term, we can estimate $H_{eq}$ by using $F \simeq 1 + f$ and integrating from a rigid bottom at $z = 0$ to a free surface at $z = H$. We find

$$H_{eq} = H \left[ \frac{1}{H \bar{\rho}(0)} \int_0^H \bar{\rho}(z) dz \right] \simeq H$$

combining this with the horizontal equations gives us the dispersion relationship for long surface gravity waves

$$\omega^2 = f^2 + g H k^2$$

and barotropic Rossby waves

$$\omega = \frac{\beta k}{k^2 + f^2 / g H}$$

Generally, we deal with scales small compared to the external deformation radius $\sqrt{g H / f}$ and can just use

$$\omega = \frac{\beta k}{k^2}$$
In addition, we have a set of internal modes. If we use a WKB approximation so that 
\( F \simeq \exp(\imath mz) \), we have
\[
gH_{eq} \simeq \frac{N^2}{m^2}
\]
and
\[
\omega^2 = \frac{f^2 m^2 + N^2 k^2}{m^2}
\]
(long internal gravity waves) or
\[
\omega = -\frac{\beta k}{k^2 + m^2 f^2/N^2}
\]
(baroclinc Rossby waves).

**Intermediate frequencies**

When the frequency is comparable to \( N \) or \( f \) and \( N^2 H << g, N^2 H^2 << \bar{\sigma}^2 \) we have
\[
gH_{eq} \simeq \frac{(N^2 - \omega^2)}{m^2}
\]
for the internal modes, leading to the full internal wave relationship
\[
\omega^2 = \frac{f^2 m^2 + N^2 k^2}{k^2 + m^2}
\]

**High frequencies**

When the frequencies are large, the vertical scales will be short and the VSE simplifies to
\[
\frac{\partial^2}{\partial z^2} F + \frac{\omega^2}{c_s^2} F = \frac{\omega^2}{g H_{eq}} F
\]
with
\[
\frac{\partial}{\partial z} F = 0 \quad \text{(FixedB)}
\]
and/or
\[
(\frac{\partial}{\partial z} - \frac{\omega^2}{g}) F = 0 \quad \text{(FreeB)}
\]
For the internal modes,
\[
gH_{eq} = \frac{\omega^2 \bar{\sigma}^2}{\omega^2 - m^2 c_s^2}
\]
and the high–frequency horizontal equation gives
\[
\omega^2 = gH_{eq} k^2 = \frac{\omega^2 \bar{\sigma}^2 k^2}{\omega^2 - m^2 c_s^2}
\]
or

\[ \omega^2 = \bar{c}_s^2 (k^2 + m^2) \]

—sound waves.

The external mode has \( F \sim \exp(mz) \) with \( m = \omega^2 / g \) from the free surface boundary condition. The equivalent depth is

\[ gH_{eq} = \frac{\omega^2 \bar{c}_s^2}{\omega^2 + m^2 \bar{c}_s^2} \]

Since these waves are still slow compared to the sound speed, this simplifies to

\[ gH_{eq} = \frac{\omega^2 \bar{c}_s^2}{m^2 \bar{c}_s^2} \]

and the horizontal equation gives

\[ \omega^2 = gH_{eq} k^2 = \frac{\omega^2 k^2}{m^2} = \frac{\omega^2 k^2 g^2}{\omega^4} \]

or

\[ \omega^2 = g|k| \]

the dispersion relationship for short gravity waves.