16. Rossby waves

We have seen that the existence of potential vorticity gradients supports the propagation of a special class of waves known as Rossby waves. These waves are the principal means by which information is transmitted through quasi-balanced flows and it is therefore fitting to examine their properties in greater depth. We begin by looking at the classical problem of barotropic Rossby wave propagation on a sphere and continue with quasi-geostrophic Rossby waves in three dimensions.

a. Barotropic Rossby waves on a sphere

The vorticity equation for barotropic disturbances to fluid at rest on a rotating sphere is

\[ \frac{d\eta}{dt} = 0, \]

(16.1)

where

\[ \eta \equiv 2\Omega \sin \varphi + \zeta. \]

Here \( \zeta \) is the relative vorticity in the \( z \) direction. Now the equation of mass continuity for two-dimensional motion on a sphere may be written

\[ \frac{1}{a} \left[ \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] = 0, \]

(16.2)
where $u$ and $v$ are the eastward and northward velocity components, $\lambda$ and $\varphi$ are longitude and latitude, and $a$ is the (mean) radius of the earth. Using (16.2) we may define a velocity streamfunction $\psi$ such that

$$u = -\frac{1}{a} \frac{\partial \psi}{\partial \varphi},$$

and

$$v = \frac{1}{a \cos \varphi} \frac{\partial \psi}{\partial \lambda}. \tag{16.3}$$

The Eulerian expansion of (16.1) can be written

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{a \cos \varphi} + v \frac{\partial \eta}{a} = 0,$$

or using (16.3),

$$\frac{\partial \eta}{\partial t} + \frac{1}{a^2 \cos \varphi} \left[ \frac{\partial \psi \partial \eta}{\partial \lambda \partial \varphi} - \frac{\partial \psi}{\partial \lambda} \frac{\partial \eta}{\partial \varphi} \right] = 0. \tag{16.4}$$

We next linearize (16.4) about the resting state ($u = v = 0$), for which $\bar{\eta} = 2\Omega \sin \varphi$, giving

$$\frac{\partial \eta'}{\partial t} + \frac{2\Omega}{a^2} \frac{\partial \psi'}{\partial \lambda} = 0, \tag{16.5}$$

where the primes denote departures from the basic state.
In spherical coordinates,
\[ \eta' = \zeta' = \hat{k} \cdot \nabla \times \mathbf{V}' \]
\[ = \frac{1}{a^2 \cos^2 \varphi \partial^2 \psi'}{\partial \lambda^2} + \cos \varphi \frac{\partial^2 \psi'}{\partial \varphi^2} \left( \cos \varphi \frac{\partial \psi'}{\partial \varphi} \right) \]  \quad \text{(16.6)}

Let’s look for modal solutions of the form
\[ \psi' = \Psi(\varphi)e^{im(\lambda-\sigma t)}, \]
where \( m \) is the zonal wavenumber and \( \sigma \) is an angular phase speed. Using this and (16.6) in (16.5) gives
\[ \frac{d^2 \Psi}{d\varphi^2} - \tan \varphi \frac{d\Psi}{d\varphi} - \left[ \frac{2\Omega}{\sigma} + \frac{m^2}{\cos^2 \varphi} \right] \Psi = 0. \]  \quad \text{(16.7)}

This can be transformed into canonical form by transforming the independent variable using
\[ \mu \equiv \sin \varphi, \]
yielding
\[ (1 - \mu^2) \frac{d^2 \Psi}{d\mu^2} - 2\mu \frac{d\Psi}{d\mu} - \left[ \frac{2\Omega}{\sigma} + \frac{m^2}{1 - \mu^2} \right] \Psi = 0. \]  \quad \text{(16.8)}

The only solutions of (16.8) that are bounded at the poles \( (\mu = \pm 1) \) have the form
\[ \Psi = A P_m^n, \]  \quad \text{(16.9)}
The first few associated Legendre functions are given in Table 16.1. The lowest order modes, for which \( m = 0 \), are zonally symmetric and have zero frequency. These are just east-west flows that do not perturb the background vorticity gradient and thus are not oscillatory. The lowest order wave mode, for which \( n = m = 1 \), has an angular frequency of \(-\Omega\) and is therefore stationary relative to absolute space. This zonal wavenumber 1 mode has maximum amplitude on the equator and decays as \( \cos \varphi \) toward the poles. Modes of greater values of \( n \) have increasingly fine meridional structure.