Dynamics of the Zonal-Mean, Time-Mean Tropical Circulation
First consider a hypothetical planet like Earth, but with no continents and no seasons and for which the only friction acting on the atmosphere is at the surface.

This planet has an exact nonlinear equilibrium solution for the flow of the atmosphere, characterized by

1. Every column is in radiative-convective equilibrium,

2. Wind vanishes at planet’s surface

3. Horizontal pressure gradients balanced by Coriolis accelerations
Hydrostatic balance:

\[ \frac{\partial p}{\partial z} = -\rho g \]

In pressure coordinates:

\[ \frac{\partial \phi}{\partial p} = -\alpha, \]

where

\[ \alpha = \frac{1}{\rho} \equiv \text{specific volume}, \]

\[ \phi = gz \equiv \text{geopotential} \]
Geostrophic balance on a sphere:

$$\left( \frac{\partial \phi}{a \partial \theta} \right)_p = -2\Omega \sin \theta u_{rel} - \frac{u_{rel}^2}{a} \tan \theta$$

Can be phrased in terms of absolute angular momentum per unit mass:

$$M = a \cos \theta \left( \Omega a \cos \theta + u \right)$$

$$\frac{\partial \phi}{\partial \theta} = -\sin \theta \left[ \frac{M^2 - \Omega^2 a^4 \cos^4 \theta}{a^2 \cos^3 \theta} \right]$$
Eliminate $\phi$ using hydrostatic equation:

$$\frac{1}{\alpha^2} \tan \theta \frac{\partial M^2}{\partial p} = \left( \frac{\partial \alpha}{\partial \theta} \right)_p = \left( \frac{\partial T}{\partial p} \right)_{s^*} \left( \frac{\partial s^*}{\partial \theta} \right)_p$$

(Thermal wind equation)

Take $s^* = s^*(\theta)$ (convective neutrality)

Integrate upward from surface ($p=p_0$), taking $u=0$ at surface:
\[ M^2 = a^2 \cos^2 \theta \left[ \Omega^2 a^2 \cos^2 \theta - \cot \theta (T_s - T) \frac{\partial s^*}{\partial \theta} \right] \]

\[ \rightarrow u^2 + 2\Omega a \cos \theta u + \cot \theta (T_s - T) \frac{\partial s^*}{\partial \theta} = 0. \]

\[ u \equiv -\frac{(T_s - T)}{2\Omega \sin \theta} \frac{\partial s^*}{\partial y} \]

Implies strongest west-east winds where entropy gradient is strongest, weighted toward equator
Two potential problems with this solution:

1. Not enough angular momentum available for required west-east wind,

2. Equilibrium solution may be unstable
How much angular momentum is needed?

At the tropopause,

\[ M^2 = a^2 \cos^2 \theta \left[ \Omega^2 a^2 \cos^2 \theta - \cot \theta (T_s - T_t) \frac{\partial s^*}{\partial \theta} \right] \]

How much angular momentum is available?

\[ M^2_{eq} = \Omega^2 a^4 \]

\[ \rightarrow \quad (T_s - T_t) \frac{\partial s^*}{\partial \theta} \geq - \frac{\Omega^2 a^2 \tan \theta (1 - \cos^4 \theta)}{\cos^2 \theta} \]
More generally, we require that angular momentum decrease away from the equator:

\[
\frac{1}{\sin \theta} \frac{\partial M^2}{\partial \theta} < 0
\]

\[
\rightarrow 4\Omega^2 a^2 \cos^3 \theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \cos^2 \theta \cot \theta (T_s - T_t) \frac{\partial s^*}{\partial \theta} \right] > 0.
\]

Let \( y = \sec^2 \theta \)

\[
\rightarrow 1 + y^3 \frac{\partial}{\partial y} \left( \frac{T_s - T_t}{\Omega^2 a^2} \frac{\partial s^*}{\partial y} \right) > 0
\]
What happens when this condition is violated?

Overturning circulation must develop

Acts to drive actual entropy distribution back toward criticality

Postulate constant angular momentum at tropopause:

$$M_t = \Omega a^2$$
Integrate thermal wind equations down from tropopause:

\[ M^2 = \Omega^2 a^4 + a^2 \cos^2 \theta \cot \theta (T - T_t) \frac{\partial s^*}{\partial \theta} \]

\[ \rightarrow u \approx \frac{\Omega a}{2} \left( \frac{1}{\cos^3 \theta} - \cos \theta \right) + \frac{T - T_t}{2\Omega a \sin \theta} \frac{\partial s^*}{\partial \theta} \]

Note: \( u \) at surface always < 0 for supercritical entropy distribution
Critical entropy distribution:

\[
\frac{T_s - T_t}{\Omega^2 a^2} \frac{\partial s^*}{\partial y} = \frac{1}{2} \left( \frac{1}{y^2} - 1 \right)
\]

\[
\rightarrow \quad \frac{T_s - T_t}{\Omega^2 a^2} s^* = \frac{T_s - T_t}{\Omega^2 a^2} s_{eq} \left( 1 - \frac{1}{2} \left( y + \frac{1}{y} \right) \right)
\]

\[
= \frac{T_s - T_t}{\Omega^2 a^2} s_{eq} \left( 1 - \frac{1}{2} \left( \cos^2 \theta + \frac{1}{\cos^2 \theta} \right) \right)
\]
Violation results in large-scale overturning circulation, known as the Hadley Circulation, that transports heat poleward and drives surface entropy gradient back toward its critical value.
Figure 2: Critical (open circles) and observed (solid line) distributions of $\theta_{es}$ for every month at the 600 mb level as a function of latitude
Simulations by Held and Hou (1980)
Zonal wind at tropopause
The cross-equatorial Hadley Circulation
Numerical simulation using a 2-D primitive equation model with parameterized convection (Pauluis, 2004)

Imposed sea surface temperature
Angular momentum and stream function, $\Delta P = 50$ mb