Question 1

Provide an example of a 2-player game with strategy set \([0, \infty)\) for either player and payoffs continuous in the strategy profile, such that no strategy survives iterated deletion of strictly dominated strategies \((S^\infty = \emptyset)\), but the set of strategies remaining at every stage is nonempty \((S^k \neq \emptyset \text{ for } k = 1, 2, \ldots)\).

Question 2

In the normal form game below player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. We only indicate player 3’s payoff. Show that action \(D\) is not a best response for player 3 to any independent belief about opponents’ play (mixed strategy for players 1 and 2), but that \(D\) is not strictly dominated. Comment.

Question 3

Each of two players \(i = 1, 2\) receives a ticket with a number drawn from a finite set \(\Theta_i\). The number written on a player’s ticket represents the size of a prize he may receive. The two prizes are drawn independently, with the value on \(i\)’s ticket distributed according to \(F_i\). Each player is asked simultaneously (and independently) whether he wants to exchange
his ticket for the other player’s ticket. If both players agree then the prizes are exchanged; otherwise each player receives his own prize. Find all Bayesian Nash equilibria (in pure or mixed strategies).

**Question 4**

A game $G = (N, S, u)$ is said to be symmetric if $S_1 = S_2 = \cdots = S_n$ and there is some function $f : S_1 \times S_1^{n-1} \to \mathbb{R}$ such that $f(s_i, s_{-i})$ is symmetric with respect to the entries in $s_{-i}$, and $u_i(s) = f(s_i, s_{-i})$ for every player $i$.

1. Consider a symmetric game $G = (N, S, u)$ in which $S_1$ is a compact and convex subset of a Euclidean space and $u_i$ is continuous and quasiconcave in $s_i$. Show that there exists a symmetric pure-strategy Nash equilibrium (i.e. a pure-strategy Nash equilibrium where every player uses the same strategy).

2. Suggest a definition for symmetric Bayesian games, $G = (N, A, \Theta, u, T, p)$, and find broad conditions on such a game $G$ that ensure that $G$ has a symmetric Bayesian Nash equilibrium.

3. Consider a Cournot oligopoly with inverse-demand function $P$ and a cost function $\gamma$ that is common to all firms. Each firm’s cost depends on its production level and its idiosyncratic cost parameter, which is drawn from a finite set $C$. Assume the vector of cost parameters $(c_1, \ldots, c_n)$ is symmetrically distributed. Each firm $i$ privately knows its own cost $c_i$, but not the others’ costs, and independently chooses a quantity $q_i$ to produce. Find conditions on $P$ and $\gamma$ that guarantee existence of a symmetric Bayesian Nash equilibrium in this game. (Note that the profit of each firm $i$ is $q_i P (q_1 + \cdots + q_n) - \gamma (q_i, c_i)$.)

**Question 5**

Let $N = \{0, 1, \ldots, n\}^2$ be a two dimensional grid. Say that two points $(x, y)$ and $(x', y')$ in $N$ are neighbors if $|x - x'| + |y - y'| = 1$. At each point $i \in N$, there is a firm, also denoted by $i$. As in a Cournot oligopoly, simultaneously, each firm $i$ chooses a quantity $q_i \in [0, 1]$ to
produce at zero marginal cost, and sells at price

\[ P_i(\theta, q, \alpha) = \theta - q_i - \sum_{k=1}^{\infty} \alpha^{k-1} \left( \sum_{j \in N_i^k} q_j / |N_i^k| \right)^k. \]

Here, \( \theta \in [1, 2] \) is a common demand parameter, and \( \alpha \in [0, 1) \) is an interaction parameter with respect to distant neighbors. \( N_i^k \) is the \( k \)-th iterated set of neighbors of \( i \): thus \( N_i^1 \) is the immediate neighbors of \( i \) (e.g., \( N_{(0,0)}^1 = \{(1,0),(0,1)\} \)), \( N_i^2 \) is the neighbors of neighbors of \( i \) (e.g., \( N_{(0,0)}^2 = \{(0,0),(0,2),(2,0),(1,1)\} \)), and so on. The payoff of firm \( i \) is its profit: \( q_i P_i \).

The value of \( \alpha \) is common knowledge, but \( \theta \) is unknown, drawn from some finite set \( \Theta \subseteq [1, 2] \). The players’ information about \( \theta \) is represented by a finite type space \( T \), with some joint prior \( p \in \Delta(\Theta \times T) \).

(1) For any choice of a Bayesian Nash equilibrium \( q^*_\alpha : T \rightarrow [0,1]^N \) of the above Bayesian game (for each \( \alpha \)), and for any \( t_i \in T_i \), find \( \lim_{\alpha \rightarrow 0} q^*_\alpha (t_i) \).

[It suffices to find a formula that consists of iterated expectations of the form

\[ E_{ij_1...j_k} [\theta | t_i] = E \left[ E \left[ \cdots E \left[ \theta | t_{j_k} \right] \cdots | t_{j_1} \right] | t_i \right], \]

where \( i, j_1, \ldots, j_k \in N \). Your formula does not need to be in closed form, but it should not refer to \( q^* \).]

(2) Simplify your result in part (a) under the assumption that \( E_{ij} [\theta | t_i] = E [\theta | t_i] \) for all \( i, j \), and \( t_i \).