Problem 1
The joint pdf of \( X \) and \( Y \) will be equal to the product of the marginal pdfs, since \( X \) and \( Y \) are independent.

\[
f_{X,Y}(x,y) = f_X(x)f_Y(y)
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}
\]
\[
= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}
\]

The transformation into polar coordinates is

\[
r^2 = X^2 + Y^2
\]
\[
\tan \theta = \frac{Y}{X}
\]

with inverse transformations

\[
X = r \cos \theta
\]
\[
Y = r \sin \theta
\]

This yeilds the following matrix of partial derivatives.

\[
\begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix}
\]

The determinant of this matrix, the Jacobian, is

\[
J = \cos \theta (r \cos \theta) - \sin \theta (-r \sin \theta)
\]
\[
= r \cos^2 \theta + r \sin^2 \theta
\]
\[
= r (\cos^2 \theta + \sin^2 \theta) = r
\]

The transformations are unique, so we can use the 1-step method without modification.

\[
f_{r\theta}(r, \theta) = r \frac{1}{2\pi} e^{-\frac{1}{2}r^2}
\]

where \( r \) lies within \([0, \infty]\) and \( \theta \) lies within \([0, 2\pi]\). Because the ranges are not dependent and the joint pdf is separable, \( r \) and \( \theta \) are also independent.

Problem 2
a. For a single random variable: \( P(X_i \leq 115) = P\left(\frac{X_i - \mu}{\sigma} \leq \frac{115 - \mu}{\sigma}\right)\).

Notice that \( Z_i = \frac{X_i - \mu}{\sigma} \) is distributed standard normal \((Z_i \sim N(0,1))\) so:
\[ P(X_i \leq 115) = P\left( \frac{Z_i}{\sqrt{225}} \leq \frac{115 - 100}{\sqrt{225}} \right) = P(Z_i \leq 1). \] Using the Table you can find that this probability is approximately equal to: 0.8413. By independence: \[ P(X_1 \leq 115, X_2 \leq 115, X_3 \leq 115, X_4 \leq 115) = P(X_1 \leq 115) P(X_2 \leq 115) P(X_3 \leq 115) P(X_4 \leq 115) = 0.8413^4 = 0.50096. \]

b. \( \bar{X}_n = \frac{\sum_{i=1}^{n} \frac{1}{n} X_i}{n} \sim N\left( \frac{\sum_{i=1}^{n} \frac{1}{n} \mu_i; \sum_{i=1}^{n} \left( \frac{1}{n} \right)^2 \sigma_i^2} \right) = N\left( 100, \frac{225}{n} \right) \) so \( \bar{X}_4 \sim N\left( 100, \left( \frac{15}{2} \right)^2 \right) \).

Thus: \( Z = \frac{\bar{X}_4 - 100}{\frac{15}{\sqrt{2}}} \) is a standard normal random variable: \( P(\bar{X}_4 < 115) = P\left( \bar{X}_4 < 115 \right) = P\left( Z < 2 \right) = 0.9772. \)

c. \[ P\left( \left| \bar{X}_n - \mu \right| \leq 5 \right) = P\left( \left| \bar{X}_n - \mu \right| \leq \frac{5}{\frac{15}{\sqrt{2}}} \right) = P\left( \frac{-\sqrt{n}}{3} \leq Z \leq \frac{\sqrt{n}}{3} \right) = 0.95. \]

From the table we know that: \( P(Z \leq 1.96) \approx 0.975 \) and using the symmetry of the normal distribution this implies that \( P(-1.96 \leq Z \leq 1.96) \approx 0.95, \) so \( \frac{\sqrt{n}}{3} = 1.96 \Rightarrow n = (1.96 \cdot 3)^2 = 34.574. \) We want the smallest integer and it is \( n_0 = 35. \)

**Problem 3**

a. The number of heads \( (H) \) in 10 independent flips of a fair coin is distributed \( Binomial\left( 10, \frac{1}{2} \right) \).

\[ P(0 \leq H \leq 4) = \sum_{k=0}^{4} \binom{10}{k} \left( \frac{1}{2} \right)^k \left( \frac{1}{2} \right)^{10-k} = \sum_{k=0}^{4} \binom{10}{k} \left( \frac{1}{2} \right)^{10} = \left( \frac{1}{2} \right)^{10} \left( \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} \right) = \frac{386}{1024} = 0.37695 \]

b. Since \( H \) is binomial we can calculate its mean and variance: \( E[H] = 10 \cdot (0.5) = 5, Var[H] = 10 \cdot (0.5) (1 - 0.5) = 2.5. \) The approximation relies on the assumption that \( H \) is distributed similar to a normal random variable, so:

\[ \frac{H - E[H]}{\sqrt{Var[H]}} = \frac{H - 5}{\sqrt{2.5}} \sim Z \sim N(0, 1). \]

Therefore: \( P(0 \leq H \leq 4) = \]

\[ P\left( \frac{0 - E[H]}{\sqrt{Var[H]}} \leq \frac{H - E[H]}{\sqrt{Var[H]}} \leq \frac{4 - E[H]}{\sqrt{Var[H]}} \right) \approx P\left( \frac{-5}{\sqrt{2.5}} \leq Z \leq \frac{1}{\sqrt{2.5}} \right) \approx P(-3.162 \leq Z \leq -0.632) = \]

\( P(Z \leq 3.162) - P(Z \leq 0.632) \approx 0.999 - 0.736 = 0.263. \) Thus the approximation is not very accurate for \( n = 10. \)

c. Now \( P(0 \leq H \leq 40) = P\left( \frac{0 - E[H]}{\sqrt{Var[H]}} \leq \frac{H - E[H]}{\sqrt{Var[H]}} \leq \frac{40 - E[H]}{\sqrt{Var[H]}} \right) \approx P\left( \frac{-50}{\sqrt{250}} \leq Z \leq \frac{-10}{\sqrt{250}} \right) = \]

\( P(-10 \leq Z \leq -2) = P(Z \leq 10) - P(Z \leq 2) \approx 1 - 0.977 \approx 0.023, \) which is quite close to the exact probability.

d. Exact calculation: \( P(H = 6) = \binom{100}{6} \left( \frac{1}{2} \right)^6 \left( 1 - \frac{1}{2} \right)^{100-6} = 0.15. \)

Approximation: as \( n \to \infty, p \to 0, np \to \lambda \) the binomial distribution converges to the Poisson distribution with parameter \( \lambda. \) Since here \( np = \]
5 we can approximate the distribution with a Poisson distribution where \( \lambda = 5 \): 

\[
P(H = 40) \approx \frac{e^{-\lambda} \lambda^{40}}{40!} = \frac{e^{-5} 5^{40}}{40!} = 0.146.
\]

Clearly, this is a good approximation.

**Problem 4**

First of all, to have valid pdfs, we must use 

\[
f_{X_i}(x) = \frac{1}{\sqrt{2\pi\sigma_i}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}.
\]

As always, sorry about the typo (I omitted the negative sign).

a. Because \( X_1 \) and \( X_2 \) are independent, the joint pdf is again the product of the marginal pdfs:

\[
f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2} \left( \frac{x_1-\mu_1}{\sigma_1}^2 + \frac{x_2-\mu_2}{\sigma_2}^2 \right)}
\]

b. We will use \( Y_1 \) for \( Y \). We start with the transformations

\[
Y_1 = X_1 + X_2 \\
Y_2 = X_1 - X_2
\]

which will yield the following inverse transformations:

\[
X_1 = \frac{Y_1 + Y_2}{2} \\
X_2 = \frac{Y_1 - Y_2}{2}
\]

Then the matrix of partial derivatives is

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{bmatrix}
\]

So the Jacobian is \(|\left(\frac{1}{2} \quad -\frac{1}{2}\right) \left(\frac{1}{2} \quad \frac{1}{2}\right)| = \frac{1}{2}\)
The transformation is unique, so we can use the 1-step method without modification.

\[
\begin{align*}
&f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2\pi \sigma_1 \sigma_2} e^{-\frac{1}{2} \left( \left( \frac{y_1 + y_2 - \mu_1}{\sigma_1} \right)^2 + \left( \frac{y_1 - y_2 - \mu_2}{\sigma_2} \right)^2 \right)} \\
&= \frac{1}{4\pi \sigma_1 \sigma_2} e^{-\frac{1}{2\sigma_1^2} \left( \sigma_1^2 \left( \frac{y_1 + y_2 - \mu_1}{\sigma_1} \right)^2 + \sigma_1^2 \left( \frac{y_1 - y_2 - \mu_2}{\sigma_2} \right)^2 \right)} \\
&= \frac{1}{4\pi \sigma_1 \sigma_2} e^{-\frac{1}{2\sigma_2^2} \left( \sigma_2^2 \left( y_1^2 + 2y_1 y_2 + y_2^2 - 4\mu_1 y_1 - 4\mu_1 y_2 + 4\mu_2^2 \right) + \sigma_1^2 \left( y_1^2 - 2y_1 y_2 + y_2^2 - 4\mu_2 y_1 - 4\mu_2 y_2 + 4\mu_2^2 \right) \right)} \\
&= \frac{1}{4\pi \sigma_1 \sigma_2} e^{-\left( \frac{\sigma_1^2 + \sigma_2^2}{2} \right) \left( (y_1 - (\mu_1 + \mu_2))^2 + (y_2 - (\mu_1 - \mu_2))^2 \right) - \frac{2y_1 y_2 - 2\mu_1 y_1 - 2\mu_2 y_2 + 2\mu_1^2 - 2\mu_2^2}{\sigma_1^2} - \frac{-2y_1 y_2 + 2\mu_1 y_1 + 2\mu_2 y_2 + 2\mu_1^2 - 2\mu_2^2}{\sigma_2^2}} dy_2
\end{align*}
\]

To get the pdf of \( Y \), we must integrate over \( Y_2 \).

\[
\begin{align*}
f_Y(y_1) &= \int_{-\infty}^{\infty} \frac{1}{4\pi \sigma_1 \sigma_2} e^{-\frac{1}{2\sigma_1^2} \left( (y_1 - (\mu_1 + \mu_2))^2 + (y_2 - (\mu_1 - \mu_2))^2 \right)} dy_2 \\
&= \frac{1}{4\pi \sigma_1 \sigma_2} e^{-\left( \frac{\sigma_1^2 + \sigma_2^2}{2} \right) (y_1 - (\mu_1 + \mu_2))^2} \int_{-\infty}^{\infty} e^{-\left( \frac{\sigma_1^2 + \sigma_2^2}{2} \right) (y_2 - (\mu_1 - \mu_2))^2} dy_2 \\
&= \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\left( \frac{(y_1 - (\mu_1 + \mu_2))^2}{2\sigma_1^2} \right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_2} e^{-\left( \frac{(y_2 - (\mu_1 - \mu_2))^2}{2\sigma_2^2} \right)} dy_2 \\
&= \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\left( \frac{(y_1 - (\mu_1 + \mu_2))^2}{2\sigma_1^2} \right)}
\end{align*}
\]

which has no closed form, in general. By other methods, it can be proved that \( Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \). We can show this here if we let \( \sigma_1 = \sigma_2 = \sigma \).

\[
\begin{align*}
f_Y(y_1) &= \frac{1}{4\pi \sigma^2} e^{-\left( \frac{2\sigma^2}{8\sigma^4} \right) (y_1 - (\mu_1 + \mu_2))^2} \int_{-\infty}^{\infty} e^{-\left( \frac{2\sigma^2}{8\sigma^4} \right) (y_2 - (\mu_1 - \mu_2))^2} dy_2 \\
&= \frac{1}{\sqrt{\pi} 2\sigma} e^{-\left( \frac{(y_1 - (\mu_1 + \mu_2))^2}{4\sigma^2} \right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi} 2\sigma} e^{-\left( \frac{(y_2 - (\mu_1 - \mu_2))^2}{4\sigma^2} \right)} dy_2 \\
&= \frac{1}{\sqrt{\pi} 2\sigma} e^{-\left( \frac{(y_1 - (\mu_1 + \mu_2))^2}{4\sigma^2} \right)}
\end{align*}
\]

because \( \int_{-\infty}^{\infty} e^{-\left( \frac{(y_2 - (\mu_1 - \mu_2))^2}{4\sigma^2} \right)} dy_2 \) is a standard normal, and must integrate to 1.
c. \( E(Y) = E(X_1 + X_2) = \mu_1 + \mu_2 \), since \( X_1 \) and \( X_2 \) are independent, normally distributed random variables. Similarly, \( V(Y) = V(X_1 + X_2) = \sigma_1^2 + \sigma_2^2 \) (since \( X_1 \) and \( X_2 \) are independent, their covariance is zero).

**Problem 5**

a. \( X \) is distributed \( \chi^2 \) with \( p \) degrees of freedom, so its pdf is

\[
f(x) = \frac{1}{\Gamma\left(\frac{p}{2}\right)2^{\frac{p}{2}}}x^{\frac{p}{2}-1}e^{-\frac{x}{2}}
\]

A gamma distribution for a random variable \( Y \) is of the form

\[
f(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-\frac{y}{\beta}}
\]

You can see that if we let \( \alpha = \frac{p}{2} \) and \( \beta = 2 \), \( X \) has a gamma distribution.

b. We learned in class that the square of a standard normal random variable has a \( \chi^2 \) distribution with one degree of freedom. Thus \( \left(\frac{Y-\mu}{\sigma}\right)^2 \sim \chi^2(1) \). In addition, we learned that the sum of two independent \( \chi^2 \) variables will also have a \( \chi^2 \) distribution, with degrees of freedom equal to the sum of the degrees of freedom of initial random variables. Because \( Y \) and \( X \) are independent, \( Y^2 \) and \( X \) will also be independent, and we can apply this property to conclude that \( \left(\frac{Y-\mu}{\sigma}\right)^2 + X^2 \sim \chi^2(p+1) \).

c. If \( p = 4 \), we use the fourth row of the table given in class, and look for the column corresponding to \( \alpha = 0.05 \). We can see that \( A = 9.488 \).