I. Arbitrary Impedance Terminations

\[ v(z = 0, t) = v_L(t) = \text{Re} \left[ \hat{V}_L e^{j \omega t} \right] \]

\[ i(z = 0, t) = i_L(t) = \text{Re} \left[ \hat{I}_L e^{j \omega t} \right] , \hat{I}_L = \frac{\hat{V}_L}{Z_L} \]

\[ \hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz} \]

\[ \hat{i}(z) = Y_0 \left[ \hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz} \right] \]

\[ \hat{v}(z = 0) = \hat{V}_L = \hat{V}_+ + \hat{V}_- \]

\[ \hat{i}(z = 0) = \hat{I}_L = \frac{\hat{V}_L}{Z_L} = Y_0 \left[ \hat{V}_+ - \hat{V}_- \right] \implies \hat{V}_+ + \hat{V}_- = \hat{V}_L \]

Add: \( 2 \hat{V}_+ = \hat{V}_L \left[ 1 + \frac{1}{Y_0 Z_L} \right] \implies \hat{V}_+ = \frac{\hat{V}_L}{2} \left[ \frac{Y_0 Z_L + 1}{Y_0 Z_L} \right] \)

Subtract: \( 2 \hat{V}_- = \hat{V}_L \left[ 1 - \frac{1}{Y_0 Z_L} \right] \implies \hat{V}_- = \frac{\hat{V}_L}{2} \left[ \frac{Y_0 Z_L - 1}{Y_0 Z_L} \right] \)

Load reflection coefficient: \( \Gamma_L = \frac{\hat{V}_-}{\hat{V}_+} = \frac{Y_0 Z_L - 1}{Y_0 Z_L + 1} \)

\[ = \frac{Z_L - Z_0}{Z_L + Z_0} \]
Generalized reflection coefficient:

\[ \Gamma(z) = \frac{\hat{v}_- e^{+jkz}}{\hat{v}_+ e^{-jkz}} = \hat{V}_- e^{2jkz} = \Gamma_L e^{2jkz} \]

\[ \hat{v}(z) = \hat{V}_+ e^{-jkz} [1 + \Gamma(z)] \]

\[ \hat{i}(z) = Z_0 \hat{V}_+ e^{-jkz} [1 - \Gamma(z)] \]

\[ Z_n(z) = \frac{Z(z)}{Z_0} = \frac{\hat{v}(z)}{i(z)Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \]

Normalized impedance

\[ \Gamma(z) = \frac{Z_n(z) - 1}{Z_n(z) + 1} \]

Properties

A. \(|\Gamma(z)| = |\Gamma_L| \leq 1\)

B. \[
\Gamma \left( z \pm \frac{\lambda}{2} \right) = \Gamma(z) \\
Z_n \left( z \pm \frac{\lambda}{2} \right) = Z_n(z)
\]

C. \[
\Gamma \left( z \pm \frac{\lambda}{4} \right) = -\Gamma(z) \\
Z_n \left( z \pm \frac{\lambda}{4} \right) = \frac{1}{Z_n(z)} = Y_n(z) = \frac{Y(z)}{Y_0} = \frac{i(z)}{Y_0 \hat{v}(z)}
\]

D. If line is matched, \(Z_L = Z_0, \Gamma_L = 0, Z_n(z) = 1\)

II. Load Impedance Reflected Back to the Source

\[
Z_n(z = 0) = \frac{Z_L}{Z_0} = \frac{R_L + jX_L}{Z_0} \\
Z_n \left( z = -\frac{\lambda}{4} \right) = \frac{1}{Z_n(z = 0)} = \frac{Z_0}{R_L + jX_L} = \frac{Z (z = -\frac{\lambda}{4})}{Z_0}
\]

Figure 8-17  The normalized impedance reflected back through a quarter-wave-long line inverts. (a) The time-average power delivered to a complex load can be maximized if \(Y\) is adjusted to just cancel the reactive admittance of the load reflected back to the source with \(R\), equalling the resulting input resistance.

To tune the line, choose $Y = -\frac{jX_L}{Z_0} \Rightarrow Y_T = \frac{R_L}{Z_0}$.

$Y$ is usually created by a variable length short circuited transmission line called a stub. There is maximum power into the line if $R_S = \frac{Z_0^2}{R_L} \Rightarrow \langle P \rangle_{\text{max}} = \frac{1}{2} \left(\frac{V_0}{R_S}\right)^2 = \frac{1}{8} \frac{V_0^2}{R_S} = \frac{1}{8} \frac{V_0^2 R_L}{Z_0^2}$.

**III. Quarter Wavelength Matching**

(b) If the length $l_2$ of the second transmission line shown is a quarter wave long or an odd integer multiple of $\lambda/4$ and its characteristic impedance is equal to the geometric average of $Z_1$ and $R_L$, the input impedance $Z_{in}$ is matched to $Z_1$.


To match $Z_1$ to $R_L \Rightarrow Z_1 = \frac{Z_2^2}{R_L} = \frac{Z_2^2}{R_L}, Z_2 = \sqrt{Z_1 R_L}$. 
IV. Smith Chart

\[ Z_n(z) = r + jx \quad \Gamma(z) = \Gamma_r + j\Gamma_i \]
\[ Z_n(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} \Rightarrow r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} \]
\[ r = \frac{1-\Gamma^2_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \quad x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \]

1. \( \left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \frac{1}{(1+r)^2} \) Orthogonal
2. \( (\Gamma_r - 1)^2 + (\Gamma_i - \frac{1}{2})^2 = \frac{1}{x^2} \) Circles

1. Circle of radius \( \frac{1}{1+r} \) Center at \( \Gamma_i = 0, \Gamma_r = \frac{r}{1+r} \)
2. Circle of radius \( \frac{1}{|x|} \) Center at \( \Gamma_i = \frac{1}{x}, \Gamma_r = 1 \)

Figure 8-18 For passive loads the Smith chart is constructed on the complex \( \Gamma \) plane. (a) Circles of constant normal 1 are constructed with the centers and radii shown. Various values of \( r \) and \( x \) are shown.

Figure 8-19  A complete Smith chart.

\[ i(t) = |I| \sin(\omega t - \phi) \]

\[ |I| = \frac{V_0}{|50 + Z(z = -l)|} \]

\[ \phi = \tan^{-1} \left( \frac{\text{Im}[Z(z = -l)]}{50 + \text{Re}[Z(z = -l)]} \right) \]

V. Standing Wave Parameters

\[ \hat{v}(z) = \hat{V}_+ e^{-jkz} [1 + \Gamma(z)] \quad \Rightarrow \quad |\hat{v}(z)| = |\hat{V}_+||1 + \Gamma(z)| \]
\[ \hat{i}(z) = Y_0 \hat{V}_+ e^{-jkz} [1 - \Gamma(z)] \quad \Rightarrow \quad |\hat{i}(z)| = Y_0|\hat{V}_+||1 - \Gamma(z)| \]

\[ \frac{|\hat{v}(z)|_{\text{max}}}{|\hat{v}(z)|_{\text{min}}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \text{VSWR (Voltage Standing Wave Ratio)} \]
\[ |\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \]
\[ \Gamma_L = |\Gamma_L| e^{j\phi} \]
\[ 2kd_{\text{min}} = \phi + \pi \Rightarrow \phi = \pi \left( \frac{4d_{\text{min}}}{\lambda} - 1 \right) \]

\( d_{\text{min}} \) is the shortest distance from load to first voltage minimum (at B in Figure 8-21 above)

Special Cases:

A. Matched Line: \( \Gamma_L = 0 \), VSWR = 1
B. Short or open circuited line: \( |\Gamma_L| = 1 \), VSWR = \( \infty \)
C. \( \frac{|\hat{v}(z)|_{\text{peak}}}{V_+} = 1 + |\Gamma_L|; \frac{|\hat{v}(z)|_{\text{min}}}{V_+} = 1 - |\Gamma_L| \)
D. \( \frac{|\hat{i}(z=0)|}{V_+} = |1 + \Gamma_L|; \frac{|\hat{i}(z=0)|}{\gamma_0 V_+} = |1 - \Gamma_L| \)
E. If \( Z_L = R_L \) (real), then \( \Gamma_L \) real. If \( Z_L > Z_0 \), VSWR = \( \frac{Z_L}{Z_0} \). If \( Z_L < Z_0 \), VSWR = \( \frac{Z_0}{Z_L} \)
Load Impedance: 

\[ Z_L = Z_0 \frac{1 + |\Gamma_L|e^{j\phi}}{1 - |\Gamma_L|e^{j\phi}} \]

\[ = Z_0 \left[ \frac{VSWR + 1 + (VSWR - 1)e^{j\phi}}{VSWR + 1 - (VSWR - 1)e^{j\phi}} \right] \]

\[ = Z_0 \left[ \frac{VSWR - j \tan \left( \frac{\phi}{2} \right)}{1 - jVSWR \tan \left( \frac{\phi}{2} \right)} \right] \]

\[ = Z_0 \left[ \frac{1 - jVSWR \tan (kd_{\min})}{VSWR - j \tan (kd_{\min})} \right] \]

Example: \( Z_0 = 50 \Omega, VSWR = 2 \)

\( d = \) distance between successive voltage minima
\[ d_{\text{min}} = \text{distance from load to first minimum} \]
\[ = 10 \text{ cm} \]
\[ \lambda = 2d = 80 \text{ cm} \Rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8} = 2.5\pi \]
\[ d_{\text{min}} = 10 \text{ cm} = 0.1 \text{ m} \Rightarrow kd_{\text{min}} = 2.5\pi(0.1) = \frac{\pi}{4} \]
\[ |\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{1}{3} \]
\[ \phi = \frac{4d_{\text{min}}}{\lambda} - 1 = \frac{4(0.1)}{0.8} - 1 = -\frac{1}{2} \Rightarrow \phi = -\frac{\pi}{2} \]
\[ \Gamma_L = |\Gamma_L|e^{j\phi} = \frac{1}{3}e^{-j\pi/2} = -\frac{j}{3} \]
\[ Z_L = Z_0 \left[ 1 - j\text{VSWR tan}(kd_{\text{min}}) \right] \]
\[ = \frac{50 \left[ 1 - j(2)\tan\left(\frac{\pi}{4}\right) \right]}{2 - j\tan\left(\frac{\pi}{4}\right)} \]
\[ = \frac{50(1 - 2j)}{2 - j} = 40 - 30j \text{ ohms} \]