6.241 Dynamic Systems and Control
Lecture 13: I/O Stability
Readings: DDV, Chapters 15, 16

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\( \mathcal{L}_2 \)-induced norm

**Theorem (\( \mathcal{H}_\infty \) norm is the \( \mathcal{L}_2 \)-induced norm)**

The \( \mathcal{L}_2 \)-induced norm of a causal, CT, LTI, stable system \( S \) with impulse response \( H(t) \) and transfer function \( H(s) \) is

\[
\| S \|_{2, \text{ind}} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[H(j\omega)] = \| H \|_\infty.
\]

- From Parseval’s equality, \( \| y \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y'(j\omega)Y(j\omega) \, d\omega. \)

- Hence,

\[
\| y \|_2^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}} \sigma_{\max}(H(j\omega))^2 U'(j\omega)U(j\omega) \, d\omega \leq \sup_{\omega} \sigma_{\max}[H(j\omega)]^2 \| u \|_2^2.
\]

- To show the bound is tight, pick (SISO case) \( u(t) = \exp(\epsilon t + j\omega_0 t) \), i.e., \( U(s) = 1/(s - \epsilon - j\omega_0) \), with \( \epsilon < 0 \). Then, \( \| y \|_2^2 = |H(\epsilon + j\omega_0)|^2 \| u \|_2^2 \)

- As \( \epsilon \to 0 \), by the continuity of \( H \) on the imaginary axis, the gain approaches \( |H(j\omega_0)| \).
Computation of $\mathcal{H}_\infty$ norm

**Theorem**

Let $H(s) = C(sI - A)^{-1}B$ be the transfer function of a stable, strictly causal ($D = 0$) LTI system. Define

$$M_\gamma = \begin{bmatrix}
A & \frac{1}{\gamma}BB^T \\
-\frac{1}{\gamma}C^T C & -A^T
\end{bmatrix}.$$

Then $\|H\|_\infty < \gamma$ if and only if $M_\gamma$ has no purely imaginary eigenvalues.

- This allows using bisections to compute $\|H\|_\infty$ to arbitrary precision.

- Similar formulas exist for the general case ($D \neq 0$), but are more complicated.
Computation of $\mathcal{H}_\infty$ norm

[diagram with $H(s)$ and $H^T(-s)$ in unit positive feedback]

- $\|H\| < \gamma$ if and only if $I - \frac{1}{\gamma^2} H'(j\omega)H(j\omega)$ is invertible for all $\omega \in \mathbb{R}$, i.e., if and only if $G_\gamma(s) = \left[I - \frac{1}{\gamma^2} H^T(-s)H(s)\right]^{-1}$ has no poles on the imaginary axis.

- The next step is to build a realization of $G_\gamma(s)$.

- $H^T(-s) = -B^T(sl + A)^{-T}C^T$, so a realization of this is $(-A^T, -C^T, B^T, 0)$.

- Putting together the realizations, and eliminating the internal variables, one gets the system matrix of the realization we seek as

$$M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -C^TC & -A^T \end{bmatrix}.$$
Energy of the impulse response

- Consider a stable, strictly causal CT LTI system with state-space model \((A, B, C, 0)\).

- The energy of the response to an unit impulse can be computed as

\[
\|H\|^2_{L_2} = \text{Tr} \left[ \int_0^{+\infty} H(t)^T H(t) \, dt \right] = \frac{1}{2\pi} \text{Tr} \left[ \int_{-\infty}^{+\infty} H(j\omega)' H(j\omega) \, ds \right] = \|H\|^2_{H_2},
\]

- This can be computed exactly noting that

\[
\|H\|^2_{L_2} = \text{Tr} \left[ \int_0^{+\infty} C e^{At} B B^T e^{A^T t} C^T \, dt \right] = \text{Tr} \left[ CPC^T \right],
\]

where \(P\) (called the controllability gramian) can be computed through the Lyapunov equation

\[
AP + PA^T + BB^T = 0.
\]
Some remarks on the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms$^1$

- There is no general relationship between $\mathcal{H}_\infty$ and $\mathcal{H}_\infty$.
- For example, consider

$$G_1(s) = \frac{1}{\epsilon s + 1} \quad G_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1}$$

As $\epsilon \to 0$, $\|G_1\|_\infty = \|G_2\|_\infty = 1$, but $\|G_1\|_2 \to \infty$, and $\|G_2\|_2 \to 0$.

- The $\mathcal{H}_\infty$ norm is an induced norm; then, the sub-multiplicative property holds, i.e., for any $G_1, G_2 \in \mathcal{H}_\infty$,

$$\|G_1 G_2\|_{\mathcal{H}_\infty} \leq \|G_1\|_{\mathcal{H}_\infty} \|G_2\|_{\mathcal{H}_\infty}$$

- The $\mathcal{H}_2$ norm is not an induced norm. So, in general, the submultiplicative property does not hold.

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$^1$John Wen, 2006
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