Problem Set 7

Problem 7.1
A stable linear system with a relay feedback excitation is modeled by

\[ \dot{x}(t) = Ax(t) + B \text{sgn}(Cx(t)), \]

where \( A \) is a Hurwitz matrix, \( B \) is a column matrix, \( C \) is a row matrix, and \( \text{sgn}(y) \) denotes the sign nonlinearity

\[ \text{sgn}(y) = \begin{cases} 
1, & y > 0, \\
0, & y = 0, \\
-1, & y < 0.
\end{cases} \]

For \( T > 0 \), a \( 2T \)-periodic solution \( x = x(t) \) of (7.1) is called a regular unimodal limit cycle if \( Cx(t) = -Cx(t + T) > 0 \) for all \( t \in (0, T) \), and \( CAx(0) > |CB| \).

(a) Derive a necessary and sufficient condition of exponential local stability of the regular unimodal limit cycle (assuming it exists and \( A, B, C, T \) are given).

(b) Use the result from (a) to find an example of system (7.1) with a Hurwitz matrix \( A \) and an unstable regular unimodal limit cycle.

Problem 7.2
A linear system controlled by modulation of its coefficients is modeled by

\[ \dot{x}(t) = (A + Bu(t))x(t), \]

where \( A, B \) are fixed \( n \)-by-\( n \) matrices, and \( u(t) \in \mathbb{R} \) is a scalar control.

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(a) Is it possible for the system to be controllable over the set of all non-zero vectors \( \bar{x} \in \mathbb{R}^n, \bar{x} \neq 0 \), when \( n \geq 3 \)? In other words, is it possible to find matrices \( A, B \) with \( n > 2 \) such that for every non-zero \( \bar{x}_0, \bar{x}_1 \) there exist \( T > 0 \) and a bounded function \( u: [0, T] \mapsto \mathbb{R} \) such that the solution of (7.2) with \( x(0) = \bar{x}_0 \) satisfies \( x(T) = \bar{x}_1 \)?

(b) Is it possible for the system to be full state feedback linearizable in a neighborhood of some point \( \bar{x}_0 \in \mathbb{R}^n \) for some \( n > 2 \)?

**Problem 7.3**

A nonlinear ODE control model with control input \( u \) and controlled output \( y \) is defined by equations

\[
\begin{align*}
\dot{x}_1 &= x_2 + x_3^2, \\
\dot{x}_2 &= (1 - 2x_3)u + a \sin(x_1) - x_2 + x_3 - x_3^2, \\
\dot{x}_3 &= u, \\
y &= x_1,
\end{align*}
\]

where \( a \) is a real parameter.

(a) Output feedback linearize the system over a largest subset \( X_0 \) of \( \mathbb{R}^3 \).

(b) Design a (dynamical) feedback controller with inputs \( x(t), r(t) \), where \( r = r(t) \) is the reference input, such that for every bounded \( r = r(t) \) the system state \( x(t) \) stays bounded as \( t \to \infty \), and \( y(t) \to r(t) \) as \( t \to \infty \) whenever \( r = r(t) \) is constant.

(c) Find all values of \( a \in \mathbb{R} \) for which the open loop system is full state feedback linearizable.

(d) Try to design a dynamical feedback controller with inputs \( y(t), r(t) \) which achieves the objectives from (b). Test your design by a computer simulation.