Problem Set 1 Solutions\textsuperscript{1}

Problem 1.1

Behavior set $\mathcal{B}$ of an autonomous system with a scalar binary DT output consists of all DT signals $w = w(t) \in \{0, 1\}$ which change value at most once for $0 \leq t < \infty$.

(a) Give an example of two signals $w_1, w_2 \in \mathcal{B}$ which commute at $t = 3$, but do not define same state of $\mathcal{B}$ at $t = 3$.

To answer this and the following questions, let us begin with formulating necessary and sufficient conditions for two signals $z_1, z_2 \in \mathcal{B}$ to commute and to define same state of $\mathcal{B}$ at a given time $t$.

For $w \in \mathcal{B}$, $t \in [0, \infty)$ let

$$w[t] = \begin{cases} \lim_{\tau \to t, \tau < t} w(\tau), & \text{if } t > 0, \\ w(0), & \text{if } t = 0 \end{cases}$$

be the left side limit value of $w$ at $t$. Let

$$N_+(w, t) = \begin{cases} 0, & \text{if } w(t) = \lim_{\tau \to \infty} w(\tau), \\ 1, & \text{otherwise} \end{cases}$$

be the number of discontinuities of $w(\tau)$ between $\tau = t$ and $\tau = \infty$. Similarly, let

$$N_-(w, t) = \begin{cases} 0, & \text{if } w(0) = w[t], \\ 1, & \text{otherwise} \end{cases}$$

be the number of discontinuities of $w(\tau)$ between $\tau = 0$ and $\tau = t - 0$.

\textsuperscript{1}Version of October 2, 2003
Lemma 1.1 Signals $z_1, z_2 \in \mathcal{B}$ commute at time $t \in [0, \infty)$ if and only if $z_1(t) = z_2(t)$ and

\[ N_-(z_1, t) + N_+(z_2, t) + |z_2(t) - z_1[t]| \leq 1 \quad (1.1) \]

and

\[ N_-(z_2, t) + N_+(z_1, t) + |z_1(t) - z_2[t]| \leq 1. \quad (1.2) \]

Proof First note that the “hybrid” signal $z_{12}$, obtained by “gluing” the past of $z_1$ (before time $t$) to the future of $z_2$ (from $t$ to $\infty$), is a discrete time signal if and only if $z_1(t) = z_2(t)$. Moreover, since the discontinuities of $z_{12}$ result from three causes: discontinuities of $z_1(\tau)$ before $\tau = t$, discontinuities of $z_2$ between $\tau = t$ and $\tau = \infty$, and the inequality between $z_1[t]$ and $z_1(t)$, condition (1.1) is necessary and sufficient for $z_{12} \in \mathcal{B}$ (subject to $z_1(t) = z_2(t)$). Similarly, considering the discontinuities of the other “hybrid” obtained by “gluing” the past of $z_2$ to the future of $z_2$ yields (1.2).

It follows immediately from Lemma 1.1 that signals $z_1, z_2 \in \mathcal{B}$ define same state of $\mathcal{B}$ at time $t \in [0, \infty)$ if and only if

\[ N_-(z_1, t) = N_-(z_2, t), \ D(z_1, t) = D(z_2, t), \ N_+(z_1, t) = N_+(z_2, t), \ z_1(t) = z_2(t), \quad (1.3) \]

where for $w \in \mathcal{B}$

\[ D(w, t) = |w(t) - w[t]| \]

is the indicator of a discontinuity at $t$.

For $k \in \mathbb{Z}_+$ let $u_k \in \mathcal{B}$ be defined by

\[ u_k(t) = \begin{cases} 0, & t < k, \\ 1, & t \geq k. \end{cases} \]

Then $u_1$ and $u_0$ commute but do not define same state of $\mathcal{B}$ at time $t = 3$.

(b) Give an example of two different signals $w_1, w_2 \in \mathcal{B}$ which define same state of $\mathcal{B}$ at $t = 4$.

$u_1$ and $u_2$.

(c) Find a time-invariant discrete-time finite state-space “difference inclusion” model for $\mathcal{B}$, i.e. find a finite set $X$ and functions $g : X \mapsto \{0, 1\}$, $f : X \mapsto S(X)$, where $S(X)$ denotes the set of all non-empty subsets of $X$, such that a sequence $w(0), w(1), w(2), \ldots$ can be obtained by sampling a signal $w \in \mathcal{B}$ if and only if there exists a sequence $x(0), x(1), x(2), \ldots$ of elements from $X$ such that

\[ x(t + 1) \in f(x(t)) \quad \text{and} \quad w(t) = g(x(t)) \quad \text{for} \quad t = 0, 1, 2, \ldots. \]
(Figuring out which pairs of signals define same state of \( \mathcal{B} \) at a given time is one possible way to arrive at a solution.)

Condition (1.3) naturally calls for \( X \) to be the set of all possible combinations

\[
x(t) = [N_-(w, t); N_+(w, t); D(w, t); w(t)].
\]

Note that not more than one of the first three components can be non-zero at a given time instance, and hence the total number of possible values of \( x(t) \) is eight, which further reduces to four at \( t = 0 \), since

\[
N_-(w, 0) = D(w, 0) = 0 \ \forall \ w \in \mathcal{B}.
\]

The dynamics of \( x(t) \) is given by

\[
f([0; 0; 0; 0]) = \{[0; 0; 0; 0]\},
\]

\[
f([0; 0; 0; 1]) = \{[0; 0; 0; 1]\},
\]

\[
f([1; 0; 0; 0]) = \{[1; 0; 0; 0]\},
\]

\[
f([1; 0; 0; 1]) = \{[1; 0; 0; 1]\},
\]

\[
f([0; 1; 0; 0]) = \{[1; 0; 0; 0]\},
\]

\[
f([0; 1; 0; 1]) = \{[1; 0; 0; 1]\},
\]

\[
f([0; 0; 1; 0]) = \{[0; 0; 1; 0], [0; 1; 0; 1]\},
\]

\[
f([0; 0; 1; 1]) = \{[0; 0; 1; 1], [0; 1; 0; 0]\},
\]

while \( g(x(t)) \) is simply the last bit of \( x(t) \).

This model is not the minimal state space model of \( \mathcal{B} \). Note that last two bits of \( x(t + 1) \), as well as \( w(t) \), depend only on the last two bits of \( x(t) \). Hence a model of \( \mathcal{B} \) with a two-bit state space \( X_* = \{0, 1\} \times \{0, 1\} \) can be given by

\[
f_*([0; 0]) = \{[0; 0]\}, \ f_*([0; 1]) = \{[0; 1]\}, \ f_*([1; 0]) = \{[0; 1], [1; 0]\}, \ f_*([1; 1]) = \{[0; 0], [1; 1]\},
\]

and

\[
g_*([x_1; x_2]) = x_2.
\]
Problem 1.2

Consider differential equation

\[ \ddot{y}(t) + \text{sgn}(\dot{y}(t) + y(t)) = 0. \]

(a) Write down an equivalent ODE \( \dot{x}(t) = a(x(t)) \) for the state vector \( x(t) = [y(t); \dot{y}(t)] \).

\[
a \left( \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \right) = \left[ \begin{array}{c} \bar{x}_2 \\ -\text{sgn}(\bar{x}_1 + \bar{x}_2) \end{array} \right].
\]

(b) Find all vectors \( \bar{x}_0 \in \mathbb{R}^2 \) for which the ODE from (a) does not have a solution \( x : [t_0, t_1] \mapsto \mathbb{R}^2 \) (with \( t_1 > t_0 \)) satisfying initial condition \( x(t_0) = x_0 \).

Solutions (forward in time) do not exist for

\[
\bar{x}_0 \in X_0 = \left\{ \left[ \begin{array}{c} \bar{x}_{01} \\ \bar{x}_{02} \end{array} \right] \in \mathbb{R}^2 : \bar{x}_{01} + \bar{x}_{02} = 0, \bar{x}_{01} \in [-1, 1], \bar{x}_{01} \neq 0 \right\}.
\]

To show this, note first that, for

\[ \bar{x}_{01} + \bar{x}_{02} \geq 0, \bar{x}_{02} > 1, \]

a solution is given by

\[
x(t) = \left[ \begin{array}{c} \bar{x}_{01} + t\bar{x}_{02} - t^2/2 \\ \bar{x}_{02} - t \end{array} \right], \; t \in [0, 2(\bar{x}_{02} - 1)].
\]

Similarly, for

\[ \bar{x}_{01} + \bar{x}_{02} \leq 0, \bar{x}_{02} < -1, \]

a solution is given by

\[
x(t) = \left[ \begin{array}{c} \bar{x}_{01} + t\bar{x}_{02} + t^2/2 \\ \bar{x}_{02} + t \end{array} \right], \; t \in [0, 2(-\bar{x}_{02} - 1)].
\]

Finally, for \( \bar{x}_0 = 0 \) there is the equilibrium solution \( x(t) \equiv 0 \).

Now it is left to prove that no solutions with \( x(0) \in X_0 \) exist. Assume that, to the contrary, \( x : [0, \epsilon] \mapsto \mathbb{R} \) is a solution with \( \epsilon > 0 \) and \( x(0) = [-t, t] \) for some \( t \in [-1, 1], t \neq 0 \). Without loss of generality, assume that \( 0 < t \leq 1 \).

Since \( x \) is continuous, there exist \( \delta \in (0, \epsilon) \) such that \( x_2(t) > 0 \) for all \( t \in [0, \delta] \). Let \( t_0 \) be the argument of minimum of \( x_1(t) + x_2(t) \) for \( t \in [0, \delta] \). If \( x_1(t_0) + x_2(t_0) < 0 \) then \( x_2(t) - \text{sgn}(x_1(t) + x_2(t)) \geq 1 \) for \( t \) in a neighborhood of \( t_0 \), which contradicts
the assumption that \( t_0 \) is an argument of a minimum. Hence \( x_1(t) + x_2(t) \geq 0 \) for all 
\( t \in [0, \delta] \). Moreover, since \( x_1 \) is an integral of \( x_2 > 0 \), \( x_1(t) \) is strictly monotonically non-increasing on \([0, \delta]\), and hence \( x_1(t) > -1 \) for all \( t \in (0, \delta) \).

Let \( t_0 \) be the argument of maximum of \( x_1(t) + x_2(t) \) on \([0, \delta]\). If \( x_1(t_0) + x_2(t_0) > 0 \) 
then \( x_1(t) + x_2(t) > 0 \) in a neighborhood of \( t_0 \). Combined with \( x_1(t) > -1 \), this yields 
\[
d(t) = x_2(t) - \text{sgn}(x_1(t) + x_2(t)) < -x_1(t) - \text{sgn}(x_1(t) + x_2(t)) < 1 - 1 = 0.
\]
Since \( x_1 + x_2 \) is an integral of \( d \), this contradicts the assumption that \( t_0 \) is an 
argument of a maximum. Hence \( x_1(t) + x_2(t) = 0 \) for \( t \in [0, \delta] \), which implies that 
\( x_2(t) \) is a constant. Hence \( x_1(t) \) is a constant as well, which contradicts the strict 
monotonicity of \( x_1(t) \).

(c) Define a semicontinuous convex set-valued function \( \eta : \mathbb{R}^2 \mapsto 2^{\mathbb{R}^2} \) 
such that \( a(\bar{x}) \in \eta(\bar{x}) \) for all \( x \). Make sure the sets \( \eta(\bar{x}) \) are the 
smallest possible subject to these constraints.

First note that \( a([\bar{x}_1, \bar{x}_2]) \) converges to \([\bar{x}_2, 1]\) as \( \bar{x}_2 \to \bar{x}_2^0 \) within the open half plane 
\( \bar{x}_1 + \bar{x}_2 < 0 \). Similarly, \( a([\bar{x}_1, \bar{x}_2]) \) converges to \([\bar{x}_2, -1]\) as \( \bar{x}_2 \to \bar{x}_2^0 \) subject to 
\( \bar{x}_1 + \bar{x}_2 > 0 \). Hence one must have \( \eta(\bar{x}) \supset \eta_0(\bar{x}) \), where 
\[
\eta_0 \left( \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \right) = \left\{ \begin{bmatrix} \bar{x}_2 \\ -t \end{bmatrix} : t \in \nu(\bar{x}_1 + \bar{x}_2) \right\},
\]
\[
\nu(y) = \begin{cases} 
\{1\}, & y > 0, \\
\{-1\}, & y < 0, \\
[-1,1], & y = 0.
\end{cases}
\]

On the other hand, it is easy to check that the compact convex set-valued function 
\( \eta_0 \) is semicontinuous. Hence \( \eta = \eta_0 \).

(d) Find explicitly all solutions of the differential inclusion \( \dot{x}(t) \in \eta(x(t)) \) 
satisfying initial conditions \( x(0) = x_0 \), where \( x_0 \) are the vectors found 
in (b). Such solutions are called sliding modes.

The proof in (b) can be repeated to show that all such solutions will stay on the 
hyperplane \( x_1(t) + x_2(t) = 0 \). Hence 
\[
x_1(t) = x_1(0)e^{-t}, \quad x_2(t) = x_2(0)e^{-t}.
\]

(e) Repeat (c) for \( a : \mathbb{R}^2 \mapsto \mathbb{R}^2 \) defined by 
\[
a([x_1; x_2]) = [\text{sgn}(x_1); \text{sgn}(x_2)].
\]
\[
\eta \left( \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \right) = \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} : c_1 \in \nu(\bar{x}_1), \ c_2 \in \nu(\bar{x}_2) \right\}.
\]
Problem 1.3

For the statements below, state whether they are true or false. For true statements, give a brief proof (can refer to lecture notes or books). For false statements, give a counterexample.

(a) All maximal solutions of ODE \( \dot{x}(t) = \exp(-x(t)^2) \) are defined on the whole time axis \( \{t\} = \mathbb{R} \).

This statement is true. Indeed, a maximal solution \( x = x(t) \) is defined on an interval with a finite bound \( t_* \) only when \( |x(t)| \to \infty \) as \( t \to t_* \). However, \( x(t) \) is an integral of a function not exceeding 1 by absolute value. Hence \( |x(t) - x(t_0)| \leq |t - t_0| \) for all \( t \), and therefore \( |x(t)| \) cannot approach infinity on a finite time interval.

(b) All solutions \( x : \mathbb{R} \mapsto \mathbb{R} \) of the ODE

\[
\dot{x}(t) = \begin{cases} 
  x(t)/t, & t \neq 0, \\
  0, & t = 0 
\end{cases}
\]

are such that \( x(t) = -x(-t) \) for all \( t \in \mathbb{R} \).

This statement is false. Indeed, for every pair \( c_1, c_2 \in \mathbb{R} \) the function

\[
x(t) = \begin{cases} 
  c_1 t, & t \leq 0, \\
  c_2 t, & t > 0 
\end{cases}
\]

is a solution of the ODE, which can be verified by checking that

\[
x(t_2) - x(t_1) = \int_{t_1}^{t_2} \frac{x(t)}{t} dt \quad \forall t_1, t_2.
\]

(c) If constant signal \( w(t) \equiv 1 \) belongs to a system behavior set \( \mathcal{B} \), but constant signal \( w(t) \equiv -1 \) does not then the system is not linear.

This statement is true. Indeed, if \( \mathcal{B} \) is linear then \( cw \in \mathcal{B} \) for all \( c \in \mathbb{R}, w \in \mathcal{B} \). With \( c = -1 \) this means that, for a linear system, \( w \in \mathcal{B} \) if and only if \( -w \in \mathcal{B} \).