Energy-conserving Transducers

Joel Voldman*
Massachusetts Institute of Technology

(*with thanks to SDS)
Outline

> Last time

> The two-port capacitor as a model for energy-conserving transducers

> The transverse electrostatic actuator

> A look at pull-in

> Formulating state equations
Last time: equivalent circuits

> Learned how to describe systems as lumped elements and equivalent circuits


Image by MIT OpenCourseWare.
Last time: equivalent circuits

> Saw that lumped elements in different domains all had equivalent circuits

> Introduced generalized notation to describe many different domains

\[
\begin{align*}
e &= \frac{dp}{dt} \\
p &= p_o + \int_0^t e dt
\end{align*}
\]

\[
\begin{align*}
f &= \frac{dq}{dt} \\
q &= q_o + \int_0^t f dt
\end{align*}
\]
## Equivalent circuit elements

<table>
<thead>
<tr>
<th>General</th>
<th>Electrical</th>
<th>Mechanical</th>
<th>Fluidic</th>
<th>Thermal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort (e)</td>
<td>Voltage, V</td>
<td>Force, F</td>
<td>Pressure, P</td>
<td>Temp. diff., ΔT</td>
</tr>
<tr>
<td>Flow (f)</td>
<td>Current, I</td>
<td>Velocity, v</td>
<td>Vol. flow rate, Q</td>
<td>Heat flow, Q</td>
</tr>
<tr>
<td>Displacement (q)</td>
<td>Charge, Q</td>
<td>Displacement, x</td>
<td>Volume, V</td>
<td>Heat, Q</td>
</tr>
<tr>
<td>Momentum (p)</td>
<td>-</td>
<td>Momentum, p</td>
<td>Pressure</td>
<td>-</td>
</tr>
<tr>
<td>Resistance</td>
<td>Resistor, R</td>
<td>Damper, b</td>
<td>Fluidic resistance, R</td>
<td>Thermal resistance, R</td>
</tr>
<tr>
<td>Capacitance</td>
<td>Capacitor, C</td>
<td>Spring, k</td>
<td>Fluidic capacitance, C</td>
<td>Heat capacity, mcp</td>
</tr>
<tr>
<td>Inertance</td>
<td>Inductor, L</td>
<td>Mass, m</td>
<td>Inertance, M</td>
<td>-</td>
</tr>
<tr>
<td>Node law</td>
<td>KCL</td>
<td>Continuity of space</td>
<td>Mass conservation</td>
<td>Heat energy conservation</td>
</tr>
<tr>
<td>Mesh law</td>
<td>KVL</td>
<td>Newton’s 2nd law</td>
<td>Pressure is relative</td>
<td>Temperature is relative</td>
</tr>
</tbody>
</table>

Cite as: Joel Voldman, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Today’s goal

> How do we model an electrical force applied to the cantilever?

> How can we describe converting energy between domains?

> This leads to energy-conserving transducers
Outline

> Last time

> The two-port capacitor as a model for energy-conserving transducers

> The transverse electrostatic actuator

> A look at pull-in

> Formulating state equations
General Considerations

> In MEMS, we are often interested in sensors and actuators

> We can classify sensors and actuators by the way they handle energy:
  
  • Energy-conserving transducers
    » Examples: electrostatic, magnetostatic, and piezoelectric actuators
  
  • Transducers that use a dissipative effect
    » Examples: resistive or piezoresistive sensors

> There are fundamental reasons why these two classes must be treated differently.
  
  • Energy-conserving transducers depend only on the state variables that control energy storage. Therefore, quasi-static analysis is OK.
  
  • Dissipative transducers depend, in addition, on state variables that determine the rate of energy dissipation, and are more complex as a result.
An Energy-Conserving Transducer

> By definition, it dissipates no energy, hence contains no resistive elements in its representation

> Instead, it can store energy from different domains – this creates the transducer action

> Because the stored energy is potential energy, we use a capacitor to represent the element, but because there are both mechanical and electrical inputs, this must be a new element: a two-port capacitor
Capacitor with moveable plate

> A charged capacitor has a force of attraction between its two plates

> If one of the plates is moveable, one can make an electrostatic actuator.

Various ways of charging

> Charging at fixed gap
  • An external force is required to prevent plate motion
  • No movement → No mechanical work

> Charging at zero gap, then lifting
  • No electrical energy at zero gap
  • Must do mechanical work to lift the plate

> Either method results in stored energy

Cite as: Joel Voldman, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Charging at Fixed Gap

> The stored energy is obtained directly from the definition for a linear capacitor

> Anticipating that the gap might vary, we now explicitly include the gap as a variable that determines the stored energy

\[
W = \int_0^q edq = \int_0^Q VdQ = \int_0^Q \frac{Q}{C} dQ
\]

\[
W(Q, g) = \frac{Q^2}{2C} = \frac{Q^2 g}{2\varepsilon A}
\]

\[
C = \frac{\varepsilon A}{g}
\]
Pulling Up at Fixed Charge

> Putting charge at zero gap stores no electrical energy

\[
C \rightarrow \infty \Rightarrow W = \frac{Q^2}{2C} \rightarrow 0
\]

> Once charge is applied, determining stored energy is a mechanics problem.

> In determining the force, we must avoid double-counting of charge

\[
E = \frac{Q}{2\varepsilon A}
\]

E-field of bottom plate

\[
F = QE = \frac{Q^2}{2\varepsilon A}
\]

\[
W(Q, g) = \int_0^g Fdg = \frac{Q^2g}{2\varepsilon A}
\]

The final stored energy is same as before!
ONLY depends on Q and g, not the path!
Lossless transducers

> The energy in the system ONLY depends on the STATE variables (e.g., Q, g) and NOT how we put the energy in

- The system is lossless/conservative

\[
\frac{dW}{dt} = P_{electrical} + P_{mechanical} = VI + Fg \\
\frac{dW}{dt} = V \frac{dQ}{dt} + F \frac{dg}{dt} \\
dW = VdQ + Fdg
\]

A Differential Version

Since we can modify the stored energy either by changing the charge or moving the plate, we can think of the stored energy as defined differentially.

\[ dW = VdQ + Fdg \]

This leads to a pair of differential relations for the force and voltage.

\[ F = \left. \frac{\partial W(Q, g)}{\partial g} \right|_Q \]
\[ V = \left. \frac{\partial W(Q, g)}{\partial Q} \right|_g \]
Revisit charging the capacitor

> The energy only depends on $Q, g$

- These are thus the STATE variables for this transducer

\[ W(Q, g) = \frac{Q^2 g}{2\varepsilon A} \]
The two-port capacitor

> This transducer is what will couple our electrical domain to our mechanical domain

\[
W(Q, g) = \frac{Q^2 g}{2\varepsilon A}
\]

\[
V = \frac{\partial W(Q, g)}{\partial Q} \bigg|_g = \frac{Qg}{\varepsilon A}
\]

\[
F = \frac{\partial W(Q, g)}{\partial g} \bigg|_Q = \frac{Q^2}{2\varepsilon A}
\]

A different example

> What if the material in the gap could move?

\[ W(Q, x) = \frac{Q^2}{2C} \]

\[
C = \frac{l}{g} \left( \epsilon_0 x + \epsilon(x_0 - x) \right)
\]

\[
F = \left. \frac{\partial W(Q, x)}{\partial x} \right|_Q = \frac{Q^2 g}{2l} \frac{\partial}{\partial x} \left( \frac{1}{\epsilon_0 x + \epsilon(x_0 - x)} \right)
\]

\[
F = \frac{Q^2 g}{2l} \frac{\epsilon - \epsilon_0}{\left( \epsilon_0 x + \epsilon(x_0 - x) \right)^2}
\]
Outline

> Last time

> The two-port capacitor as a model for energy-conserving transducers

> The transverse electrostatic actuator

> A look at pull-in

> Formulating state equations
The Electrostatic Actuator

> If we now add a spring to the upper plate to supply the external mechanical force, a practical actuator results

> We are getting closer to our RF switch…


Two methods of electrical control

> Charge control

• Capacitor is charged from a current source, specifically controlling the charge regardless of the motion of the plate
• This method is analyzed with the stored energy

> Voltage control

• Capacitor is charged from a voltage source, specifically controlling the voltage regardless of the motion of the plate
• This method is analyzed with the stored co-energy
Charge control

Following the causal path

1. Current source determines the charge
2. Charge determines the force (at any gap!)
3. Force determines the extension of the spring
4. Extension of the spring determines the gap
5. Charge and gap together determine the voltage

\[ Q = \int_{0}^{t} i_{\text{in}}(t) \, dt \]

\[ F = \frac{\partial W}{\partial g} \bigg|_{g} = \frac{Q^2}{2\varepsilon A} \]

\[ z = \frac{F}{k} \]  

initial displacement

\[ g = g_0 - z \]

\[ g = g_0 - \frac{Q^2}{2\varepsilon A k} \]
Charge control

> Let’s get voltage, normalize and plot

\[
V = \left. \frac{\partial W}{\partial Q} \right|_g = \frac{Qg}{\varepsilon A} = \frac{Q \left( g_0 - \frac{Q^2}{2\varepsilon A k} \right)}{\varepsilon A}
\]

> Normalize variables to make easier to plot

- First normalize \( V \) and \( Q \) to some nominal values
- Introduce \( \xi \) (normalized displacement) that goes from 0 \( (g=g_0) \) to 1 \( (g=0) \)

\[
v = \frac{V}{V_0} \quad q = \frac{Q}{Q_0} \quad \xi = \frac{z}{g_0} = \frac{g_0 - g}{g_0}
\]

- Define \( Q_0 \) and \( V_0 \) using expression above

\[
V_0 = \frac{Q_0 g_0}{\varepsilon A} \quad Q_0^2 = 2\varepsilon A k g_0
\]
Charge control

> Now, plug in to non-dimensionalize

\[ V = \frac{Q \left( g_0 - \frac{Q^2}{2\varepsilon A k} \right)}{\varepsilon A} \]

\[ V = \frac{(qQ_0) \left( g_0 - \frac{(qQ_0)^2}{2\varepsilon A k} \right)}{\varepsilon A} = \frac{(qQ_0)(g_0 - q^2 g_0)}{\varepsilon A} \]

\[ V = \frac{Q_0 g_0}{\varepsilon A} q(1 - q^2) \Rightarrow v = q(1 - q^2) \]

\[ \xi = 1 - \frac{g}{g_0} = 1 - (1 - q^2) \Rightarrow \xi = q^2 \]

> Now we get expressions relating voltage and displacement to charge
Charge control

> Actuator is stable at all gaps – the voltage goes to zero at zero gap

> The voltage is multivalued→ the charge uniquely determines the state and thus the energy

\[
V = \left. \frac{\partial W}{\partial Q} \right|_g = \frac{Q g}{\varepsilon A} = \frac{Q \left( g_0 - \frac{Q^2}{2\varepsilon A k} \right)}{\varepsilon A}
\]

Cite as: Joel Voldman, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Co-Energy

> For voltage control, we cannot use $W(Q,g)$ directly, because we cannot maintain constant charge. Instead we use the co-energy

- So we change variables

Recall:

\[ W^*(e_1) = q_1e_1 - W(q_1) \]

\[ W^*(V, g) = QV - W(Q, g) \]

\[ dW^*(V, g) = d(QV) - dW(Q, g) \]

\[ dW^*(V, g) = [QdV + VdQ] - [VdQ + Fdg] \]

\[ \Rightarrow Q = \frac{\partial W^*(V, g)}{\partial V} \bigg|_g \]

\[ \Rightarrow F = -\frac{\partial W^*(V, g)}{\partial g} \bigg|_V \]
Voltage control

> Following the causal path

1. Voltage and gap (implicitly) determines the force
2. Force determines the spring extension
3. And thus the gap
4. Voltage and gap together determine the charge

\[
W^*(V_{in}, g) = \frac{1}{2} CV_{in}^2 = \frac{\varepsilon A}{2g} V_{in}^2
\]

\[
1) \quad F = -\frac{\partial W^*}{\partial g} \bigg|_V = \frac{\varepsilon AV_{in}^2}{2g^2}
\]

\[
2) \quad g = g_0 - z
\]
\[
\quad z = \frac{F}{k}
\]

\[
3) \quad g = g_0 - \frac{\varepsilon AV_{in}^2}{2kg^2}
\]

\[
4) \quad Q = \frac{\varepsilon A}{g} V_{in} = CV_{in}
\]

Outline

> Last time

> The two-port capacitor as a model for energy-conserving transducers

> The transverse electrostatic actuator

> A look at pull-in

> Formulating state equations
Forces and stability

> Let’s examine the net force on the actuator
\[ F_{\text{Net}} = F_{\text{mech}} - F_{\text{elec}} = 0 \]
\[ = k(g_0 - g) - \frac{\varepsilon AV^2}{2g^2} = 0 \]

positive force increases gap

> Nondimensionalize again

\[ \xi = \frac{(g_0 - g)}{g_0} \]
\[ v = \frac{V}{V_{PI}} \]
\[ V_{PI}^2 = \frac{8kg_0^3}{27\varepsilon A} \]

\[ F_{\text{Net}} = k\xi g_0 - \frac{\varepsilon Av^2}{2g^2} \frac{8kg_0^3}{27\varepsilon A} = 0 \]
\[ \xi - \frac{4v^2 g_0^2}{27g^2} = 0 \]
\[ \xi - \frac{4v^2}{27(1-\xi)^2} = 0 \]
Stability criterion

> At low voltage, there are two intersections
  • Which is stable?
> At higher voltages, there are none
  • What is happening?

The position of the actuator is stable only when there is a net restoring force when the system is disturbed from equilibrium.

![Diagram showing stable and unstable positions with net force](image-url)
Stability criterion

> We can plot the normalized NET force versus normalized gap and check

\[ F_{Net} = F_{mech} - F_{elec} \]

\[ = k(g_0 - g) - \frac{\varepsilon AV^2}{2g^2} \]

\[ 1 - \xi = \frac{g}{g_0} \]

\[ \xi = \xi - \frac{4v^2}{27(1 - \xi)^2} \]

\[ f_{net} = \left( \frac{g_0}{g} + 1 \right) - \frac{4v^2}{27\left( \frac{g_0}{g} \right)^2} \]
> So what we want is a negative slope

> In this example, this means that the spring constant must exceed a critical value that varies with voltage

\[
F_{Net} = k(g_0 - g) - \frac{\varepsilon AV^2}{2g^2}
\]

Stability:
\[
\frac{\partial F_{Net}}{\partial g} = \left(-k + \frac{\varepsilon AV^2}{g^3}\right) < 0
\]
\[
\frac{\varepsilon AV^2}{g^3} < k
\]
Stability criterion

> If the voltage is too large, the system becomes unstable, and we encounter pull-in.

> Right at pull-in, the spring constant is at the critical value AND static equilibrium is maintained.

At pull-in:

\[ k = \frac{\varepsilon AV_{pl}^2}{g_{pl}^3} \]

\[ k(g_0 - g_{pl}) = \frac{\varepsilon AV_{pl}^2}{2g_{pl}^2} \]

\[ V_{pl} = \sqrt{\frac{8kg_0^3}{27\varepsilon A}} \]

\[ g_{pl} = \frac{2}{3} g_0 \]
Stability analysis of pull-in

> Plot normalized gap versus normalized voltage

> Solve cubic equation

\[
g = g_0 - \frac{\varepsilon A V_{in}^2}{2k g^2}
\]

In Matlab:

\[
g = \text{fzero}(@(g)(g - g0 + \varepsilon A V^2/(2k g^2)),g0);
\]
Release voltage after pull-in

> After pull-in less voltage is needed to keep beam down

> Find force when pulled down

> Equate to mechanical force to get hold-down voltage

> Is usually much less than pull-in voltage

\[
F_{elec}|_{g=\delta} = \frac{\varepsilon AV_{in}^2}{2\delta^2}
\]
\[
F_{mech}|_{g=\delta} = k(g_0 - \delta) \approx k\delta
\]

\[
\frac{\varepsilon AV_{HD}^2}{2\delta^2} = k\delta
\]
\[
V_{HD}^2 = \frac{2\delta^2 k\delta}{\varepsilon A}
\]

Normalize to \(V_{PI}\)

\[
V_{PI}^2 = \frac{8kg_0^3}{27\varepsilon A}
\]

\[
\left(\frac{V_{HD}}{V_{PI}}\right)^2 = \frac{27}{4}\left(\frac{\delta}{g_0}\right)^2 < 1
\]

\[
V^2 = \frac{8kg_0^3}{27\varepsilon A}
\]
Macro pull-in?

> Can we do a macroscopic pull-in demo?

> Use soft spring $k = 1 \text{ N/m}$

> Use

• $A = 8.5'' \times 11''$ plates
• $g_0 = 1 \text{ cm}$

$$V_{PI} = \frac{\sqrt{8kg_0^3}}{27\varepsilon A}$$

$$= \frac{8(1)(0.01)^3}{27\left(8.85\times10^{-12}\right)\left(8.5\times11\times(0.0254)^2\right)}$$

$$\approx 750 \text{ V}$$

> Not easy… this is why pull-in is a MEMS-specific phenomenon
Outline

> Last time

> The two-port capacitor as a model for energy-conserving transducers

> The transverse electrostatic actuator

> A look at pull-in

> Formulating state equations
Adding dynamics

> Add components to complete the system:
  • Source resistor for the voltage source
  • Inertial mass, dashpot
> This is now our RF switch!
> System is nonlinear, so we can’t use Laplace to get transfer functions
> Instead, model with state equations

![Image](image_url)


Electrical domain  |  Mechanical domain
The System is Now General

> The addition of the source resistor breaks up the distinction between voltage-controlled and charge-controlled actuation:

• For small $R$, the system behaves like a voltage-controlled actuator

• For large $R$, the system behaves like a charge-controlled actuator at short times because the “impedance” of the rest of the circuit is negligible $\Rightarrow$ the voltage source delivers a constant current $V/R^*$

*See, for example, Castaner and Senturia, JMEMS, 8, 290 (1999)
State Equations

> Dynamic equations for general system (linear or nonlinear) can be formulated by solving equivalent circuit

> In general, there is one state variable for each independent energy-storage element (port)

> Good choices for state variables: the charge on a capacitor (displacement) and the current in an inductor (momentum)

> For electrostatic transducer, need three state variables
  • Two for transducer \((Q,g)\)
  • One for mass \((dg/dt)\)

Goal:

\[
\frac{d}{dt} \begin{bmatrix} Q \\ g \\ \dot{g} \end{bmatrix} = \begin{pmatrix} \text{functions of} \\ Q, g, \dot{g} \text{ or constants} \end{pmatrix}
\]
Formulating state equations

> Start with $Q$

> We know that $\frac{dQ}{dt} = I$

> Find relation between $I$ and state variables and constants

\[
\frac{dQ}{dt} = I = \frac{1}{R} \left( V_{in} - V \right)
\]

KVL: $V_{in} - e_R - V = 0$

\[
e_R = IR
\]

\[
V_{in} - IR - V = 0
\]

\[
V = \frac{Qg}{\varepsilon A}
\]
Formulating state equations

> Now we’ll do \( \dot{g} \) 

> We know that \( \frac{dg}{dt} = \ddot{g} \)

KVL:

\[
F - e_k - e_m - e_b = 0 \quad e_k = kz
\]

\[
F - kz - m\ddot{z} - b\dot{z} = 0 \quad e_m = m\ddot{z}
\]

\[
z = g_0 - g \Rightarrow \dot{z} = -\ddot{g}, \dddot{z} = -\dddot{g}
\]

\[
F - k(g_0 - g) + mg + b\ddot{g} = 0
\]

\[
\dddot{g} = -\frac{1}{m} \left[ F - k(g_0 - g) + b\ddot{g} \right]
\]

\[
\frac{dg}{dt} = -\frac{1}{m} \left[ \frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\ddot{g} \right]
\]
Formulating state equations

> State equation for $g$ is easy: \[
\frac{dg}{dt} = \dot{g}
\]

> Collect all three nonlinear state equations

\[
\begin{bmatrix}
\frac{d}{dt}Q \\
\frac{d}{dt}g \\
\frac{d}{dt}\dot{g}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{R} \left( V_{in} - \frac{Qg}{\varepsilon A} \right) \\
\dot{g} \\
-\frac{1}{m} \left[ \frac{Q^2}{2\varepsilon A} - k(g_0 - g) + b\dot{g} \right]
\end{bmatrix}
\]

> Now we are ready to simulate dynamics (WED)
What have we wrought?

> **We have modeled a complex multi-domain 3D structure using**

  - Equivalent circuits
  - A two-port nonlinear capacitor

> **What can we now get**

  - Actuation voltage: $V_{PI}$
  - Tip dynamics

> **What have we lost**

  - Capacitor plates are not really parallel during actuation
  - Neglected fringing fields
  - Neglected stiction forces when beam is pulled in

---

Figure 9 on p. 17 in: Nguyen, C. T.-C. 
"Vibrating RF MEMS Overview: Applications to Wireless Communications."


Images removed due to copyright restrictions.
Conclusions

> We can successfully model nonlinear transducers with a new element: the two-port capacitor

> Know when to use energy or co-energy for forces
  • At best a sign error
  • At worst just wrong

> Under charge control, transverse electrostatic actuator is well-behaved

> Under voltage control, exhibits pull-in